22 Using exact values

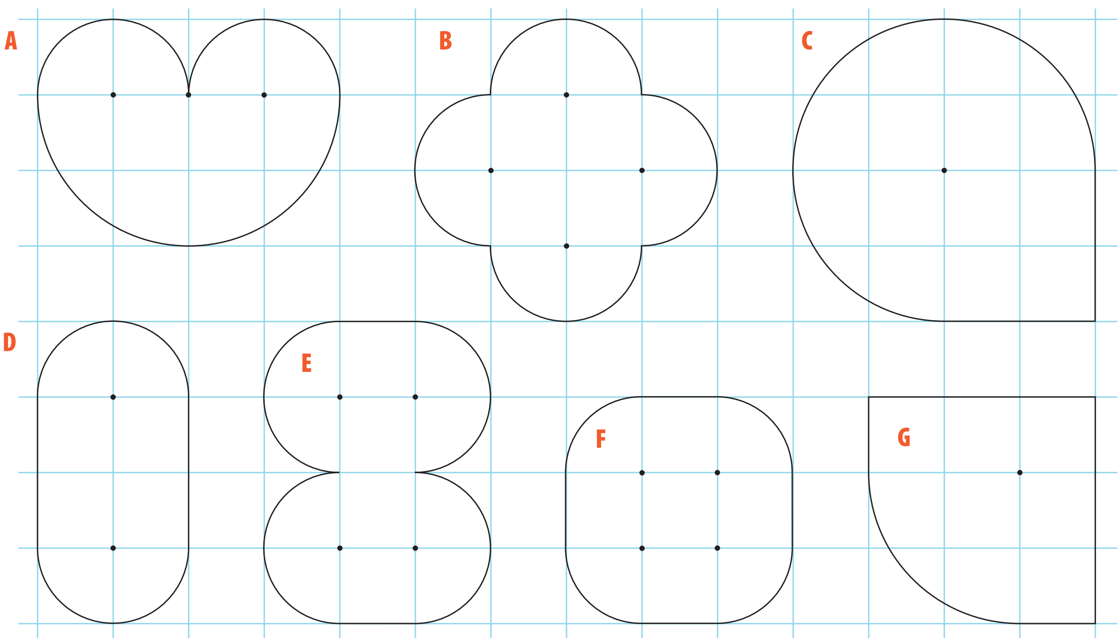
You will revise

- **finding the circumference and area of a circle**
- **using Pythagoras's theorem**

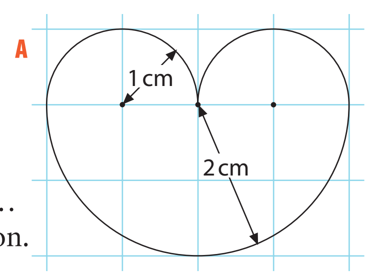
This work will help you write exact answers using terms like $\sqrt{10}$ (surds) and π, instead of changing them to approximate decimals.

A Showing π in a result

These shapes are drawn on a grid of centimetre squares.



Shape A is made of two semicircles with radius 1 cm and one semicircle with radius 2 cm. Their centres are marked with dots.



The length in centimetres of the large semicircle is

$$\tfrac{1}{2} \times 2\pi r = \tfrac{1}{2} \times 2 \times \pi \times 2 = 2\pi$$

Trying to work 2π out gives a decimal that starts as 6.283 185 307…
But the decimal doesn't end or recur. It isn't equal to an exact fraction.
So the only way to write this length in centimetres **exactly** is 2π.

A1 **(a)** Find the exact length of one small semicircle in shape A.

(b) Find the exact value of the whole perimeter of shape A.

A2 Find the exact perimeter of each of the other shapes above.

A3 **(a)** Find the area of one small semicircle in shape A on the opposite page. Give the exact answer, with π left in it.

(b) Find the exact area of the large semicircle in shape A.

(c) Find the exact value of the whole area of shape A.

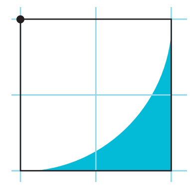
A4 Find the exact area of all the other shapes on the opposite page.

A5 **(a)** What is the area of the square with the black outline?

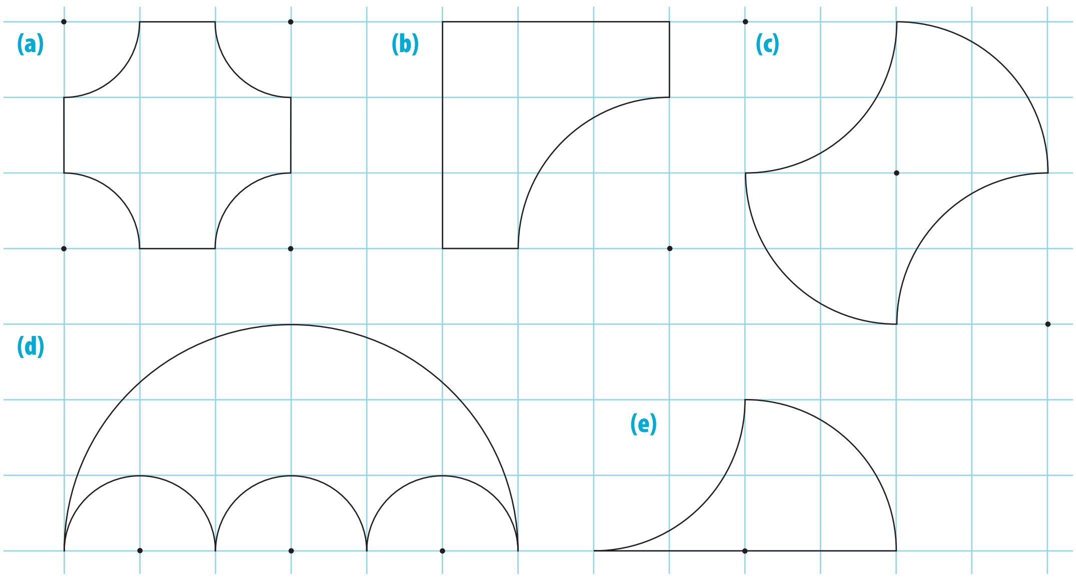
(b) What is the exact area of the quarter circle?

(c) Write an exact expression for the area of the blue shape.

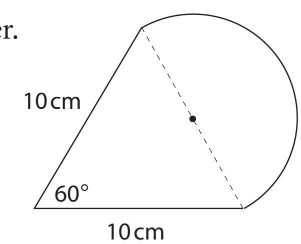
(d) Write an exact expression for the perimeter of the blue shape.



A6 For each of these shapes, find **(i)** the exact area **(ii)** the exact perimeter



A7 Find the perimeter of this shape, keeping π in your answer.



***A8** Sketch shapes with these values.

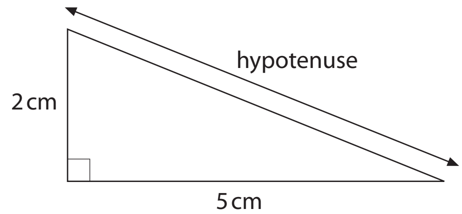
	Perimeter (cm)	Area (cm^2)
(a)	2π	2
(b)	2π	$4 - \pi$
(c)	4π	$\pi + 4$

B Showing a surd in a result

This is a right-angled triangle.

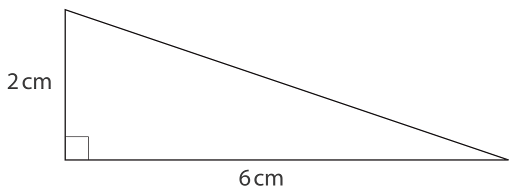
Using Pythagoras's theorem, the length in centimetres of its hypotenuse is $\sqrt{29}$.

Trying to work this out gives a decimal that starts as 5.385 164 807…
But the decimal doesn't end or recur.
So the only way to write this length in centimetres **exactly** is $\sqrt{29}$.

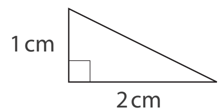


B1 Find the length of each hypotenuse, leaving each answer as an exact value.

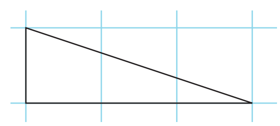
(a)



(b)



This triangle is drawn on centimetre squares.
In centimetres, the exact length of its hypotenuse is $\sqrt{10}$.
So its perimeter is $1 + 3 + \sqrt{10}$, which you can leave as the exact value $4 + \sqrt{10}$.

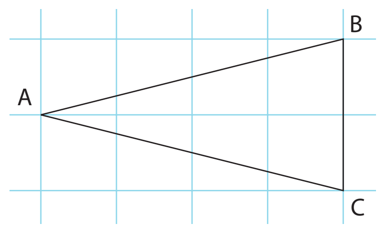


B2 This triangle is on centimetres squares.

(a) How long is AB exactly?

(b) How long is AC exactly?

(c) What is the exact perimeter of the triangle?



An expression that contains an exact square root, like $4 + \sqrt{10}$, is called a **surd**.
The answers to B1 and B2 are also surds.

Pythagoras spiral

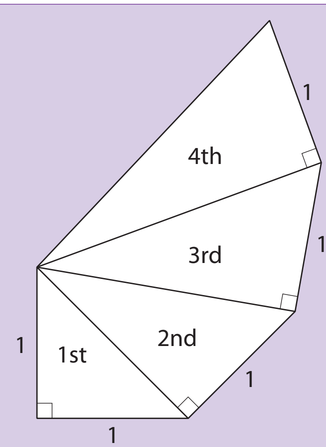
How long exactly is the hypotenuse of the first triangle?

Show that the hypotenuse of the second triangle is $\sqrt{3}$ units long.

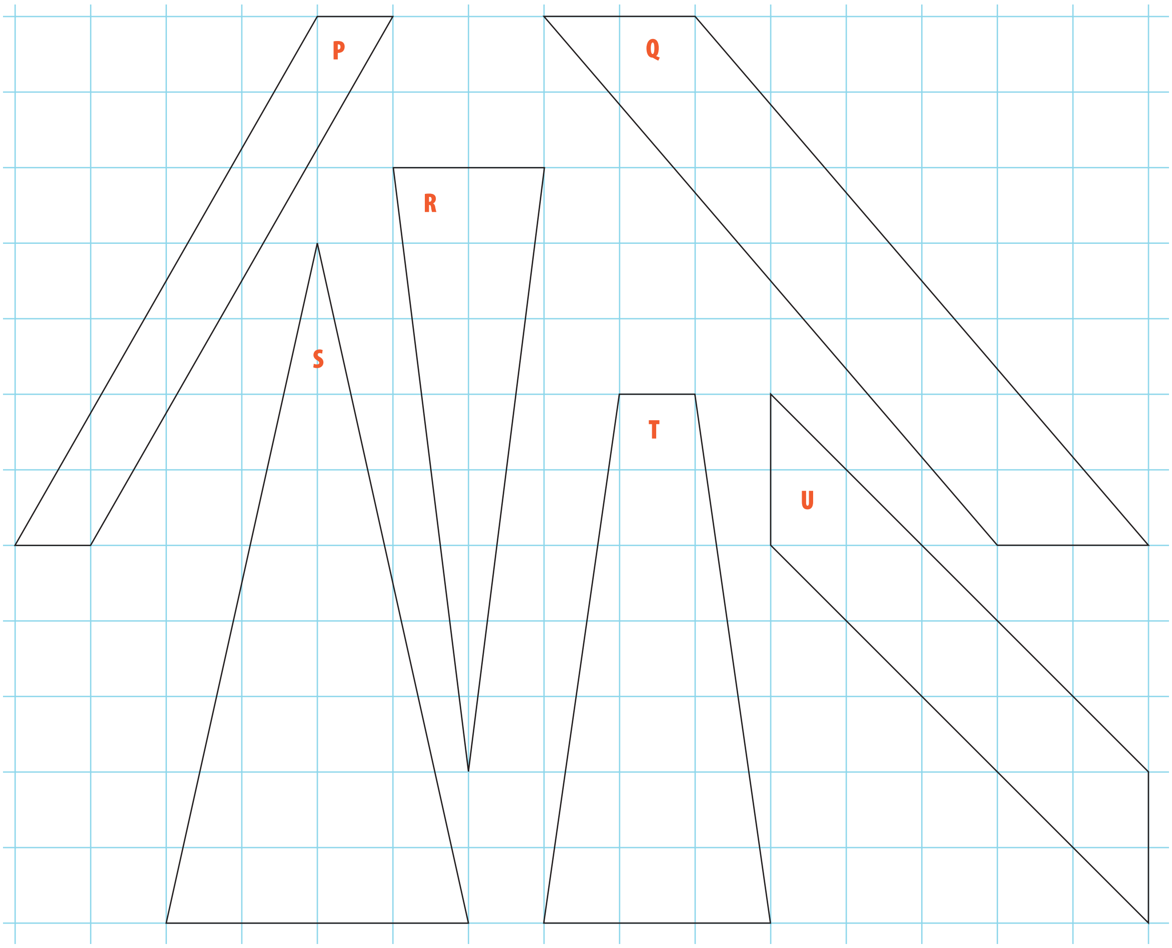
How long exactly is the hypotenuse of the fourth triangle?

If triangles continue to be added in the same way, how long exactly will be the hypotenuse of the nth triangle?

Which triangle will have a hypotenuse exactly 3 units long?

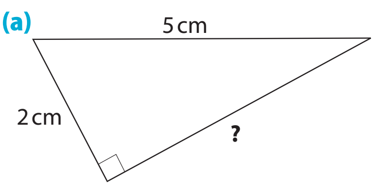


B3 Shapes P to U are drawn on a grid of centimetre squares.
By working out exact perimeters (showing your answers as surds), sort them into pairs with the same perimeter.

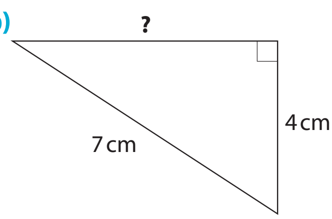


B4 Use Pythagoras to find the missing side in each of these.
They are not drawn accurately.
Leave each answer as an exact value.

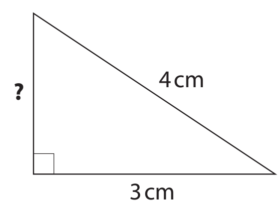
(a)



(b)



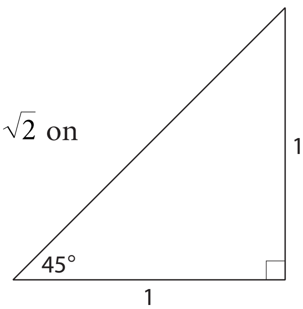
(c)



B5 Give the perimeter of each of the triangles in B4, showing your answers as surds.

B6 Calculate the area of each of the triangles in B4, showing your answers as exact values.

B7 The length of the hypotenuse of this right-angled triangle is $\sqrt{2}$.
So the sine of 45° is exactly $\frac{1}{\sqrt{2}}$.

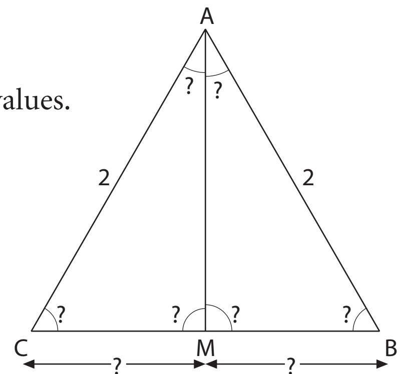


As an approximate check, make sure you can get 0.707… for $1 \div \sqrt{2}$ on a calculator, and that the calculator also gives 0.707… for sin 45°.

(a) Use the triangle to give an exact value for cos 45°.
Do an approximate calculator check for your answer.

(b) Give exact values for **(i)** sin 135° **(ii)** cos 135°

B8 This equilateral triangle has sides 2 units long.
M is the mid-point of side BC.



(a) Sketch the triangle replacing the question marks by values.

(b) How long is the line AM exactly (left as a surd)?
Mark this on your sketch.

(c) Use the diagram to give these exactly.

(i) sin 60° **(ii)** cos 60° **(iii)** tan 60°

(iv) sin 30° **(v)** cos 30° **(vi)** tan 30°

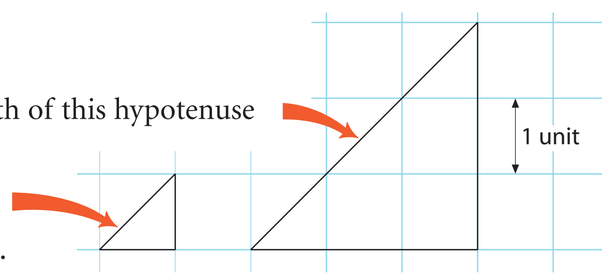
Do a calculator check as in B7 for these answers.

(d) Give these exactly.

(i) sin 120° **(ii)** cos 120° **(iii)** tan 120° **(iv)** sin 150° **(v)** cos 150° **(vi)** tan 150°

(e) Calculate the area of the equilateral triangle, leaving any surds in your answer.

Using Pythagoras, the length of this hypotenuse is $\sqrt{3^2 + 3^2} = \sqrt{18}$ units.
However this hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$ units long.



The large triangle is an enlargement with scale factor 3 of the small one, so $\sqrt{18}$ must be equal to $3\sqrt{2}$.

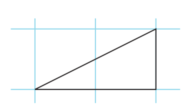
We can also show that $3\sqrt{2}$ is the square root of 18 by squaring it to get 18:

$(3\sqrt{2})^2 = (3\sqrt{2}) \times (3\sqrt{2}) = 3 \times \sqrt{2} \times 3 \times \sqrt{2} = 3 \times 3 \times \sqrt{2} \times \sqrt{2} = 9 \times 2 = 18$

We can also start with $\sqrt{18}$ and get $3\sqrt{2}$ like this: $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$.

Another example is: $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$.

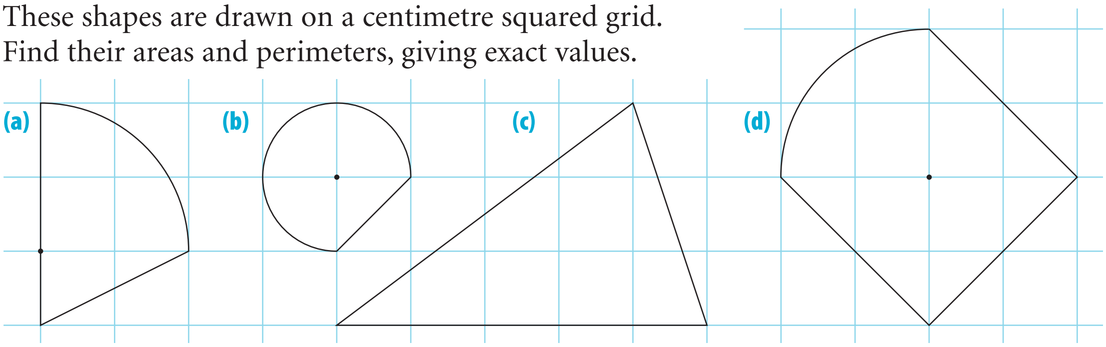
B9 Use this triangle and a suitable enlargement of it to show that $\sqrt{20} = 2\sqrt{5}$.



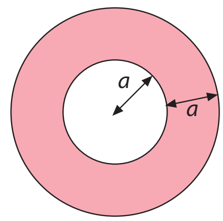
B10 Find a suitable triangle and an enlargement of it to show that $\sqrt{160} = 4\sqrt{10}$.

C Mixed questions

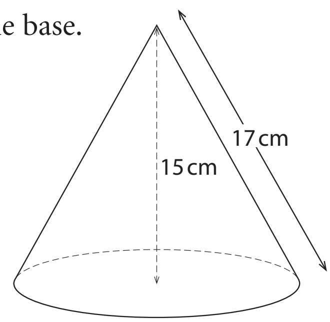
C1 These shapes are drawn on a centimetre squared grid.
Find their areas and perimeters, giving exact values.



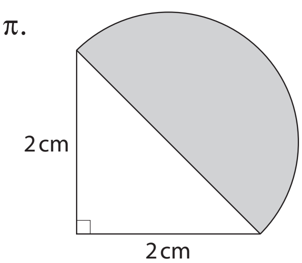
C2 The pink area is formed by two circles with the same centre.
Find the pink area exactly, in terms of a.



C3 Calculate the surface area of this cone, including the base.
Give your answer in terms of π.



C4 Find the area of the shaded semicircle in terms of π.

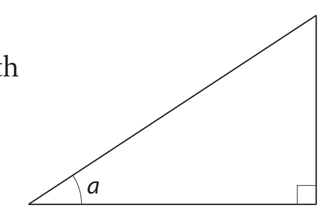


C5 A circle has an area of 10 square units. Find its radius in terms of π.

C6 In this right-angled triangle, $\cos a = \frac{3}{4}$.

(a) Make a sketch of the triangle and label all its sides with possible exact lengths.

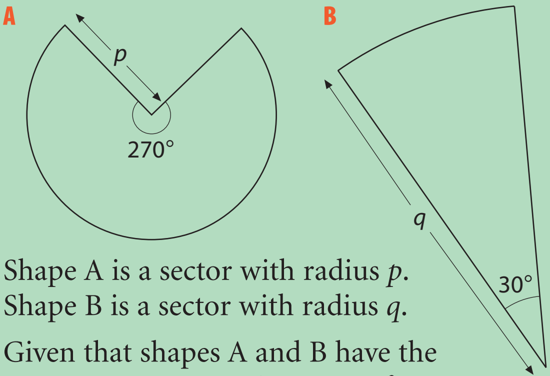
(b) Write as surds **(i)** $\sin a$ **(ii)** $\tan a$



D Exact relationships between variables

Exact expressions for quantities can be used to find relationships between variables.

Example



Shape A is a sector with radius p.
Shape B is a sector with radius q.

Given that shapes A and B have the same area, express q in terms of p.

Area of shape A = $\frac{3}{4}\pi p^2$

Area of shape B = $\frac{1}{12}\pi q^2$

Since the areas of the two shapes are equal,

$$\frac{3}{4}\pi p^2 = \frac{1}{12}\pi q^2$$

Multiplying both sides by 12,

$$9\pi p^2 = \pi q^2$$

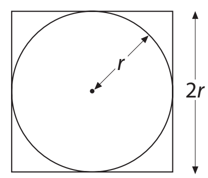
Dividing both sides by π,

$$9p^2 = q^2$$

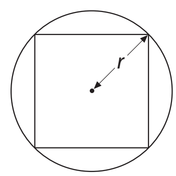
Taking the positive square root of both sides,

$$3p = q\text{, so } q = 3p$$

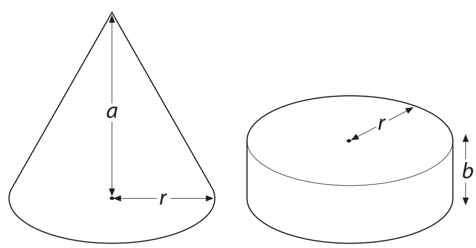
D1 **(a)** A circle of radius r fits exactly inside a square of side $2r$.
Exactly what fraction of the square does the circle occupy?



(b) A square fits exactly inside a circle of radius r.
Exactly what fraction of the circle does the square occupy?



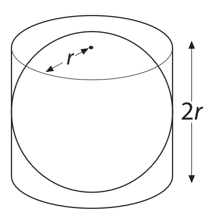
D2 The cone and cylinder have the same radius, r, and the same volume.
How are the heights a and b related?



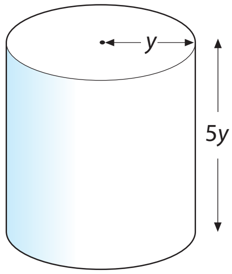
D3 A sphere has a surface area of 25π square units.

(a) What is its radius? **(b)** What is its volume, exactly?

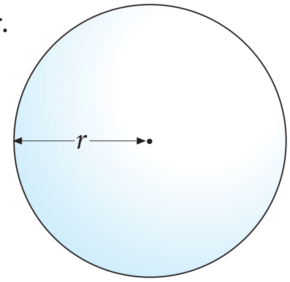
D4 A solid sphere of radius r fits inside a cylindrical space that has radius r and height $2r$.
Exactly what fraction of the cylinder's volume does the sphere occupy?



D5 A cylinder has radius y and height $5y$.



A sphere has radius r.



The total surface area of the cylinder is equal to the surface area of the sphere.

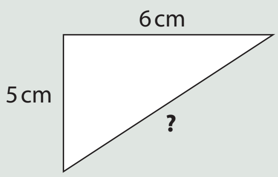
(a) Show that $r^2 = 3y^2$.

(b) Find the value of y when $r = 6$, giving your answer in the form $a\sqrt{b}$.

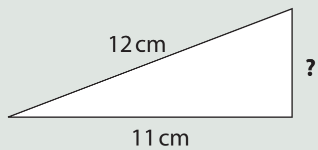
Test yourself

T1 Find the missing lengths, leaving your answers as exact values.

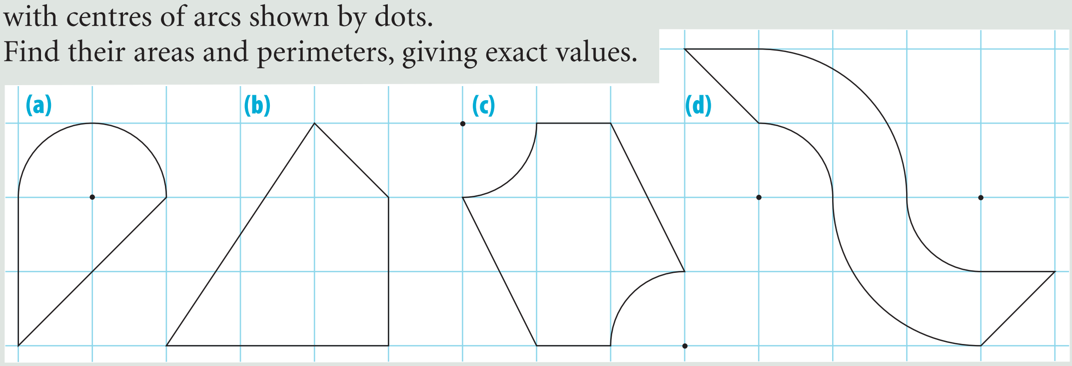
(a)



(b)

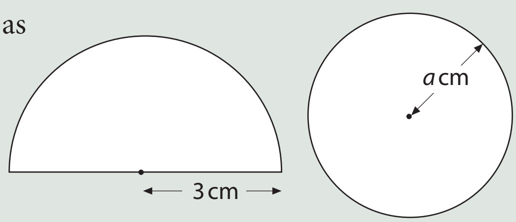


T2 These shapes are drawn on centimetre squares with centres of arcs shown by dots.
Find their areas and perimeters, giving exact values.



T3 Given that $\tan x = 3$, write exact values for $\sin x$ and $\cos x$.

T4 A semicircle of radius 3 cm has the same area as a circle of radius a cm.
Find the exact value of a.



T5 A cone has height 12 cm and volume 196 cm^3.
Find, in terms of π, the diameter of its base.