Statistics 2 for Edexcel contents

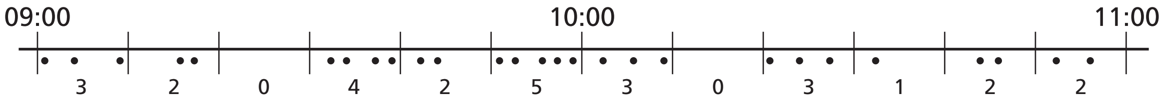
2 Poisson distribution

In this chapter you will learn
- how to calculate probabilities using the Poisson distribution
- how to find the mean and variance of a Poisson distribution
- about the additive property of the Poisson distribution

A The Poisson model (answers p 95)

An observer records the times at which cars go past on a country road.

Cars pass at random times. The times for the first two hours of observation are shown in the diagram below. The numbers below the diagram show how many cars passed during each 10-minute interval.



The number of cars passing in each 10-minute interval continued to be recorded. The complete record and a frequency table are shown here.

3 2 0 4 2 5 3 0 3 1 2 2 0 1 2 2 2 0 4 3 0 0 3 1 0

0 3 2 1 1 0 1 1 2 0 1 1 3 3 0 3 1 2 1 1 1 2 4 2 1

3 3 0 1 2 1 2 2 3 1 0 2 6 2 1 3 0 1 2 2 2 1 1 2 0

4 2 3 1 1 0 4 2 1 1 1 2 3 1 4 2 1 3 2 4 0 1 5 2 1

Number of cars in interval	Frequency
0	17
1	30
2	27
3	16
4	7
5	2
6	1
≥ 7	0
Total	100

The total number of cars is $0 \times 17 + 1 \times 30 + \ldots + 6 \times 1 = 176$.

So the mean number of cars per 10-minute interval is $\frac{176}{100} = 1.76$.

Let the discrete random variable X be the number of cars passing in a 10-minute interval.

The probability that $X = 0$ can be estimated from the data as $\frac{17}{100}$, or 0.17. Similarly for the other possible values of X.

The probability distribution for X, based on the data, is shown here.

x	0	1	2	3	4	5	6	≥ 7
$\mathrm{P}(X = x)$	0.17	0.30	0.27	0.16	0.07	0.02	0.01	0

Suppose that the mean number of cars in a 10-minute interval continues to be 1.76.

The French mathematical physicist Siméon Denis Poisson (1781–1840) suggested a mathematical model for events occurring randomly but with a constant mean rate:

$$\mathrm{P}(X = x) = \mathrm{e}^{-\lambda}\frac{\lambda^x}{x!},$$ where λ is the mean value of X (Note: 0! is taken to be 1.)

A1 By letting $\lambda = 1.76$, find the probabilities for the cars data given by the Poisson model and compare them with the values in the table above.

If events occur randomly but with a constant mean rate λ per time interval (per 10 minutes, per second, etc.), then the number X of events occurring in an interval can be modelled by the Poisson probability distribution given by

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!} \qquad x = 0, 1, 2, 3, \dots$$

λ is called the Poisson parameter. When its value is known, the distribution is completely specified.

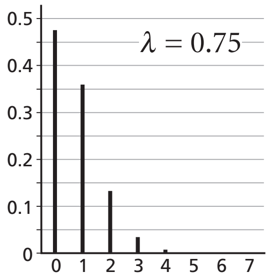
'X follows the Poisson distribution with parameter λ' is written as $X \sim \text{Po}(\lambda)$.

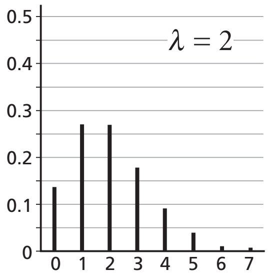
There is no limit on the possible values of X, but $P(X = x)$ gets smaller and smaller as x increases. It can be shown that the infinite sum $\sum P(X = x)$ is 1.

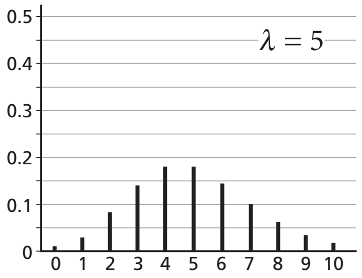
The shape of the Poisson distribution for different values of λ can be investigated by using a spreadsheet.

In Excel, e^x is denoted by EXP(x) and $x!$ by FACT(x).

Here are stick graphs for Poisson distributions with $\lambda = 0.75$, 2 and 5.







A2 Emissions from a radioactive source are detected by a Geiger counter. The emissions occur at random at a mean rate of 6 per second, so the number X of emissions in a 1-second period is such that $X \sim \text{Po}(6)$.

Calculate the probability that in a 1-second period the number of emissions is

(a) 5 **(b)** 6 **(c)** 7

The Poisson model can be used for other kinds of process involving random events.

For example when a machine produces a thread or yarn, knots may appear at random points along the length of the thread.

If the mean number of knots per metre is constant, then the number of knots appearing in a 1-metre interval has a Poisson distribution.

A3 Knots occur at random along the length of a thread with a mean rate of 3 knots per metre.

Find the probability that in a metre length of thread the number of knots is

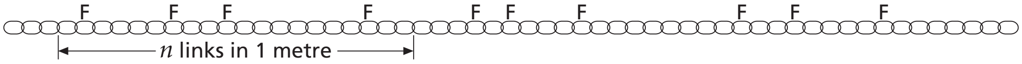
(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** more than 3

The Poisson distribution as a limiting case of the binomial distribution

The Poisson distribution can be related to the binomial distribution.

We have seen how the Poisson model can be used for knots appearing in a thread.

Imagine instead that a chain is being produced, with n links in each metre. Instead of knots, faulty links (F) may occur.



Make the following assumptions:

- The probability that a link is faulty is p and is the same for every link.
- Whether a link is faulty or not is independent of any other link.

These are the conditions for the binomial distribution to apply. The n links in a metre are the n 'trials' and the probability p that a link is faulty is the probability of 'success'.

Let X be the number of faulty links in a metre.

$\mathrm{P}(X = x)$ is given by the binomial formula, so $\mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$.

The mean value of a binomial distribution is np. Let $np = \lambda$, so that $p = \dfrac{\lambda}{n}$.

The expression for $\mathrm{P}(X = x)$ becomes $\binom{n}{x}\left(\dfrac{\lambda}{n}\right)^x\left(1-\dfrac{\lambda}{n}\right)^{n-x}$.

It can be shown that, as n gets larger and larger (so that the chain becomes more and more like a thread), this expression gets closer and closer to the Poisson probability $\mathrm{e}^{-\lambda}\dfrac{\lambda^x}{x!}$.

When n is large and p is small, the Poisson distribution with $\lambda = np$ may be used as an approximation to the binomial distribution.

Here, for example, are the first few binomial and Poisson probabilities with $n = 50$ and $p = 0.01$, so that $\lambda = np = 50 \times 0.01 = 0.5$.

	0	1	2	3
Binomial, B(50, 0.01)	0.6050	0.3056	0.0756	0.0122
Poisson, Po(0.5)	0.6065	0.3033	0.0758	0.0126

Example 1

Customers arrive at random at a checkout at the average rate of 2.4 per minute. What is the probability that in a one-minute period the number arriving is

(a) 0 **(b)** 1 **(c)** 2 or more

Solution

If the number arriving in one minute is X, then $X \sim \mathrm{Po}(2.4)$.

(a) $\mathrm{P}(X = 0) = \mathrm{e}^{-2.4} \times \dfrac{2.4^0}{0!} = 0.091$ (to 3 d.p.) **(b)** $\mathrm{P}(X = 1) = \mathrm{e}^{-2.4} \times \dfrac{2.4^1}{1!} = 0.218$ (to 3 d.p.)

(c) $\mathrm{P}(X \geq 2) = 1 - \mathrm{P}(X < 2) = 1 - (0.091 + 0.218) = 0.691$

Example 2

An icosahedral dice has 20 faces numbered from 1 to 20. By using a suitable approximation, find the probability that in 50 rolls of the dice the score 1 occurs 5 times.

Solution

Let X be the number of times score 1 occurs in 50 rolls. So $X \sim \mathrm{B}\left(50, \frac{1}{20}\right)$.
This distribution can be approximated by the Poisson distribution with $\lambda = np = 2.5$.

$\mathrm{P}(X = 5) = \mathrm{e}^{-2.5} \times \dfrac{2.5^5}{5!} = 0.0668$.

Exercise A (answers p 95)

1 Given that $X \sim \mathrm{Po}(1.5)$, calculate

(a) $\mathrm{P}(X = 0)$ **(b)** $\mathrm{P}(X = 2)$ **(c)** $\mathrm{P}(X \leq 3)$ **(d)** $\mathrm{P}(X \geq 5)$

2 Incoming calls to a call centre arrive at random but have a constant average rate of 4.2 calls in a five-minute period. Find the probability that in a given five-minute period there are

(a) no calls **(b)** exactly 3 calls **(c)** more than 3 calls

3 A botanist is investigating the number of buttercup plants on a field.
A square frame, called a quadrat, is thrown at random on to the field and the number of buttercup plants inside is counted.
The number of buttercups found per quadrat on the field can be modelled by a Poisson distribution with mean 3.4.
Find the probability that in a particular quadrat there are

(a) no buttercup plants **(b)** more than 2 buttercup plants

4 A company makes windscreens for cars. Defects in the windscreens occur at random but at an average rate of 0.8 per windscreen. Use the Poisson distribution to find the probability that a windscreen inspected has

(a) no defects **(b)** 1 defect **(c)** more than 1 defect

5 In a large population of animals 7% are carriers of a disease. By using a suitable approximation, find the probability that in a random sample of 100 animals there are 10 carriers.

6 A proofreader is checking the pages of a book for misprints.
Misprints occur at random at a mean rate of 1.2 per page.
Find the probability that a particular page of the book

(a) is free of misprints **(b)** has 1 or 2 misprints **(c)** has more than 2 misprints

7 0.25% of the packets filled by a machine are overweight. A consignment consisting of a random sample of 200 packets is unsatisfactory if it contains more than one overweight packet. By using a suitable approximation, find the probability that a consignment is unsatisfactory.

B The Poisson table (answers p 95)

In many applications of the Poisson distribution, λ is in the range $0 < \lambda \le 10$.

Tables are produced for the **cumulative** Poisson distribution for values of λ in this range.
The table for the exam is reproduced on page 90 of this book.
It shows values of $P(X \le x)$, for each given value of λ.

The part of the table shown here includes the column for $\lambda = 1.5$.
It shows that if $X \sim \text{Po}(1.5)$, then $P(X \le 4) = 0.9814$.

To find $P(X = 4)$ we need to subtract $P(X \le 3)$ from $P(X \le 4)$:

$$\begin{aligned} P(X = 4) &= P(X \le 4) - P(X \le 3) \\ &= 0.9814 - 0.9344 = 0.0470 \end{aligned}$$

$\lambda =$	**0.5**	**1.0**	**1.5**
$x = 0$	0.6065	0.3679	0.2231
1	0.9098	0.7358	0.5578
2	0.9856	0.9197	0.8088
3	0.9982	0.9810	0.9344
4	0.9998	0.9963	0.9814
5	1.0000	0.9994	0.9955
6	1.0000	0.9999	0.9991
7	1.0000	1.0000	0.9998
8	1.0000	1.0000	1.0000

To find a probability of the form $P(X \ge x)$, for example $P(X \ge 4)$, use the fact that $P(X < 4) + P(X \ge 4) = 1$.

$$P(X < 4) = P(X \le 3) = 0.9344 \quad \text{so} \quad P(X \ge 4) = 1 - 0.9344 = 0.0656$$

B1 The discrete random variable X has the Poisson distribution with parameter 0.5.
Use the table above to find

(a) $P(X \le 2)$ **(b)** $P(X = 2)$ **(c)** $P(X > 2)$ **(d)** $P(X \ge 2)$ **(e)** $P(1 \le X \le 3)$

B2 The discrete random variable Y has the Poisson distribution with parameter 1.0.
Use the table above to find

(a) $P(Y < 4)$ **(b)** $P(Y \ge 3)$ **(c)** $P(Y > 0)$ **(d)** $P(Y = 5)$ **(e)** $P(2 \le Y \le 5)$

Example 3

The discrete random variable X has the Poisson distribution with parameter 2.5.
Use a table to find

(a) $P(X = 2)$ **(b)** $P(X > 2)$ **(c)** $P(2 \le X \le 4)$

Solution

Part of the table for $\lambda = 2.5$ is shown here.

(a) $P(X = 2) = P(X \le 2) - P(X \le 1) = 0.5438 - 0.2873 = 0.2565$

(b) $P(X > 2) = 1 - P(X \le 2) = 1 - 0.5438 = 0.4562$

(c) $P(2 \le X \le 4) = P(X \le 4) - P(X \le 1) = 0.8912 - 0.2873 = 0.6039$

$x = 1$	0.2873
2	0.5438
3	0.7576
4	0.8912

Notice that, in each column, after a certain value of x the cumulative probability is recorded as 1.0000. This is because the values in the table are given to only four decimal places.

The Poisson formula gives a value of $P(X = x)$ for every possible value of x, but for large values of x these probabilities are very small. The cumulative probability gets closer and closer to 1 without ever reaching it, but this is not apparent when the values are rounded.

Exercise B (answers p 95)

Use the table on page 90 to answer these questions.

1 The discrete random variable X has the Poisson distribution with $\lambda = 3.5$. Find

(a) $P(X \leq 5)$ **(b)** $P(X = 3)$ **(c)** $P(X \geq 3)$ **(d)** $P(X > 6)$ **(e)** $P(2 < X < 6)$

2 The number of call-outs in one hour to a breakdown repair service can be modelled by a Poisson distribution with mean 2.0. The service has three repair trucks that can be sent out. Find the probability that in any one-hour period

(a) there are 2 or fewer call-outs

(b) there are exactly 3 call-outs

(c) there are more than 3 call-outs so the service cannot answer them all

3 A horticulturist is examining the leaves of a plant for viral spots.
The number of spots on a leaf can be modelled by a Poisson distribution with mean 4.5. Find the probability that for a randomly chosen leaf

(a) there are no spots

(b) there are exactly 3 spots

(c) there are more than 5 spots

4 The number of applications for grants to an organisation can be modelled by a Poisson distribution with mean 7 per month. A maximum of 5 grants may be awarded in any one month. Find the probability that in a particular month

(a) there are exactly 5 applications

(b) there are more than 5 applications, so some have to be turned down

5 A local council is deciding whether to put a pedestrian crossing, controlled by traffic lights, at a point on a stretch of road. A survey showed that during the day cars pass by the point at random times but at a constant average rate of 6.5 per minute. Find the probability that in any given minute

(a) 5 or fewer cars pass the point

(b) more than 7 cars pass the point

6 A dice has 20 faces numbered from 1 to 20. It is rolled 200 times. By using a suitable approximation, find the probability that the number of times a score of 1 occurs is greater than 15.

7 A computer printer sometimes fails to print a sheet. The probability that this happens each time is 0.04. By using a suitable approximation, find the probability that out of 100 sheets fed into the printer

(a) all are printed

(b) all but one are printed

(c) fewer than 97 are printed

C Mean, variance and standard deviation

If the discrete random variable X has the Poisson distribution with parameter λ, then the value of λ has already been identified as the mean value of X.

The mean value of X, or $\mathrm{E}(X)$, is defined as $\sum x \times \mathrm{P}(X = x)$.

When X is $\mathrm{Po}(\lambda)$, this sum is $\sum x \mathrm{e}^{-\lambda} \frac{\lambda^x}{x!}$ and it can be proved that this is equal to λ.

It can also be proved that $\mathrm{E}(X^2) = \sum x^2 \mathrm{e}^{-\lambda} \frac{\lambda^x}{x!} = \lambda^2 + \lambda$.

From the formula $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ it follows that $\mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$.

K

If X has a Poisson distribution with parameter λ, then

$\mathrm{E}(X) = \lambda$

$\mathrm{Var}(X) = \lambda$ (so the standard deviation is $\sqrt{\lambda}$)

Exercise C (answers p 95)

1 Bacteria occur at random in samples of water with a mean rate of 4.5 per ml.

(a) Find the probability that a random sample of 1 ml of water contains no bacteria.

(b) Find the probability that a random sample of 1 ml of water contains more than 5 bacteria.

(c) Write down the variance of the number of bacteria in a random sample of 1 ml of water.

2 A machine produces a continuous length of steel wire. Kinks occur at random in the wire. The number of kinks occurring in 1 metre of wire has a Poisson distribution with parameter 6.5.

(a) Write down the mean, variance and standard deviation of the number of kinks occurring in 1 metre of wire.

(b) Find the probability that the number of kinks in 1 metre lies in the range from mean – standard deviation to mean + standard deviation.

3 A researcher noted the number of live births in the litters of a species of animal. For a very large sample of litters the following relative frequencies were found.

Number of live births	0	1	2	3	4	5	6	7	8
Relative frequency	0.14	0.23	0.26	0.15	0.10	0.06	0.03	0.02	0.01

(a) From the data in the table, find the mean and variance of the number of live births in a litter.

(b) Explain how you tell from your results in (a) that the data does not fit a Poisson distribution.

D Independent Poisson distributions

Suppose that cars going in one direction pass an observer at random times at a mean rate of 3 cars per minute. Independently, cars going in the other direction also pass at random times but at a mean rate of 2 cars per minute.

It seems intuitively obvious that cars, whichever direction they are going in, pass the observer at random times at a mean rate of 5 cars per minute.

If X is the number of cars passing in a one-minute interval in the first direction, then X is Po(3). If Y is the number passing in a one-minute interval in the opposite direction, then Y is Po(2).

If X and Y are independent (so that the value of each is unaffected by the other), it seems obvious that $X + Y$ will be Po(2 + 3), or Po(5).

It is possible to prove from the Poisson formula that this is true.
The general result is as follows:

K If X and Y are independent random variables with Poisson distributions with parameters λ_1 and λ_2 respectively, then $X + Y$ has a Poisson distribution with parameter $\lambda_1 + \lambda_2$.

This property of the Poisson distribution allows us to combine intervals in an obvious way. For example, suppose that cars pass an observer at random times at a mean rate of 3 per minute. It is intuitively obvious that the mean number of cars in an interval of 5 minutes will be 15.

This can be confirmed as follows. Let X_1, X_2, X_3, X_4 and X_5 be the numbers passing in each of the five separate minutes making up a 5-minute interval. Each of these independent variables is Po(3), so the sum $X_1 + X_2 + X_3 + X_4 + X_5$ is Po(3 + 3 + 3 + 3 + 3), or Po(15).

Example 4

Cars on a production line are checked for mechanical faults and bodywork faults. Mechanical faults occur at random at a mean rate of 0.6 per car. Bodywork faults occur at random at a mean rate of 1.4 per car. The faults occur independently of each other. Find the probability that a randomly chosen car has

(a) no mechanical faults **(b)** more than 3 faults of either type

Solution

(a) If M is the number of mechanical faults on a car then $M \sim \text{Po}(0.6)$.
$\text{P}(M = 0) = \text{e}^{-0.6} = 0.5488$

(b) If B is the number of bodywork faults on a car then $B \sim \text{Po}(1.4)$.
Let T be the total number of faults so that $T = M + B$.
Then $T \sim \text{Po}(0.6 + 1.4)$, that is Po(2).

From the cumulative Poisson distribution table, $\text{P}(T \leq 3) = 0.8571$.
The probability that there are more than 3 faults is $1 - 0.8571 = 0.1429$.

Example 5

A typist makes mistakes randomly at an average rate of 1.5 per page.
Find the probability that in an article of 4 pages

(a) there are more than 10 mistakes in total

(b) there is at least one mistake on every page

Solution

(a) If X is the number of mistakes on a page, then $X \sim \text{Po}(1.5)$.
If Y is the number of mistakes on 4 pages then $Y \sim \text{Po}(4 \times 1.5)$, that is Po(6).
From the cumulative Poisson distribution table $\text{P}(Y \leq 10) = 0.9574$.
So the probability of making more than 10 mistakes is $1 - 0.9574 = 0.0426$.

(b) The probability of at least one mistake on a single page is
$1 - \text{P}(X = 0) = 1 - 0.2231 = 0.7769$ (using $X \sim \text{Po}(1.5)$).
What happens on each page is independent of what happens on the others.
So the probability of at least one mistake on every page is $(0.7769)^4 = 0.3643$.

Checking the conditions for a Poisson model

The main condition for a Poisson model to be applicable is that events occur at random but at a constant mean rate. For events to be random, each event must be independent of any other event occurring. For example, arrivals at a cinema for a family film would not be independent as people are likely to arrive in groups.

The assumption that there is a constant mean rate needs careful checking. With cars on a stretch of road or arrivals at a supermarket, the mean rate of arrival is likely to vary at different times of day. If the Poisson model is applied, it is necessary to restrict it to a part of the day where the mean rate stays the same.

The Poisson model is a theoretical model and may not fit exactly the situation it is applied to. For example, in the Poisson distribution there is no upper limit on the possible values of the variable. In a situation such as cars passing an observer, there is a physical limit to the number of cars that can pass, so the Poisson model cannot be an exact fit.

Exercise D (answers p 95)

1 Given that $X \sim \text{Po}(2.0)$ and $Y \sim \text{Po}(2.5)$, find

(a) $\text{P}(X \leq 3)$ **(b)** $\text{P}(Y = 5)$ **(c)** $\text{P}(X + Y \leq 6)$ **(d)** $\text{P}(X + Y > 6)$

2 The vehicles arriving at a petrol station are either private cars or commercial vans. The number of cars arriving in a one-minute period can be modelled by a Poisson distribution with mean 3.5.

(a) Find the probability that more than 6 cars arrive in a one-minute period.

The number of vans arriving in a one-minute period can be modelled by a Poisson distribution with mean 1.5.

(b) Find the probability that more than 6 vehicles in total (cars or vans) arrive in a one-minute period.

3 A keyboard operator enters customers' details on computerised forms.
He makes errors at random but on average he makes 1.5 errors per form.
Find the probability that on a particular form he makes

(a) no errors **(b)** at least one error **(c)** exactly one error

On a particular morning he completes three forms. Find the probability that

(d) there is a total of more than 5 errors

(e) there is at least one error on each form

4 A building society records information about each customer and about the property (house or flat) that they live in. The customer information and the property information are keyed in by two different operators.
Each operator makes errors at random. On average the first operator makes 1.3 errors per customer and the second 1.7 errors per property.

Find the probability that there are more than 2 errors in total in the record for a particular customer and their property.

5 The number of deaths in a year from the bite of a particular insect follows a Poisson distribution with mean 2.5.
Find the probability that in a period of three years

(a) there are 10 or more deaths in total

(b) there are fewer than 3 deaths in each of the three years

6 An office has four computers that operate independently of each other.
For each computer, the number of breakdowns in a week is modelled by a Poisson distribution with mean 0.5.

(a) Find the probability that in a particular week

  (i) none of the computers breaks down

  (ii) at least one computer breaks down

(b) Find the probability that in a period of four weeks

  (i) 2 or more computer breakdowns occur

  (ii) every computer breaks down at least once

7 A student suggests using the Poisson distribution to model the number of telephone calls received per minute in an office with only one telephone.
Explain why the Poisson distribution is not appropriate in this case.

8 Each weekday an inspector examines a random sample of 100 of the pots fired in a kiln. Experience has shown that whether a pot is faulty or not is independent of other pots and that the probability that a pot is faulty is 0.02.

By using a suitable approximation, find the probability that the inspector finds

(a) at least 3 faulty pots on the first day of a week

(b) a total of at least 15 faulty pots in the week

Key points

- If events occur at random but with a constant mean rate of λ per time interval (per 10 minutes, or per second, etc.) or per object (per page, per leaf, etc.), then the number X of events occurring in an interval or object has the Poisson probability distribution given by
$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$
'X follows the Poisson distribution with mean λ' is written as $X \sim \text{Po}(\lambda)$. (p 22)
- For large n and small p, the Poisson distribution with $\lambda = np$ may be used as an approximation to the binomial distribution. (p 23)
- The mean and variance of this distribution are both equal to λ. (p 27)
- If A and B are independent random variables such that $A \sim \text{Po}(\lambda_1)$, $B \sim \text{Po}(\lambda_2)$ and $X = A + B$, then $X \sim \text{Po}(\lambda_1 + \lambda_2)$. (p 28)

Mixed questions (answers p 96)

1 A newsagent buys in 5 copies of a particular magazine each week. The demand for this magazine each week can be modelled by a Poisson distribution with mean 2.5. Find the probability that, in a particular week,

(a) nobody asks for a copy of this magazine

(b) exactly three people ask for this magazine

(c) more people ask for the magazine than the newsagent has copies

2 Two roads converge at a junction. The number X of vehicles arriving per minute from the first road is such that $X \sim \text{Po}(2.8)$. The number Y arriving per minute from the second road is such that $Y \sim \text{Po}(3.2)$. X and Y are independent.

(a) Write down the distribution of $X + Y$, the total number arriving per minute at the junction.

(b) What is the variance of $X + Y$?

3 Outbreaks of a rare disease occur at random on average once in 50 years. Find the probability that

(a) there is no outbreak in the next 100 years

(b) there is at least one outbreak in the next 200 years

(c) there is no outbreak in the next 100 years but at least one in the 100 years after that

4 Natalie shuffles an ordinary pack of 52 playing cards, picks a card, looks at it, returns it to the pack and shuffles again. She does this 50 times. By using a suitable approximation, find the probability that she picks the ace of spades at least twice.

5 To be eligible to join an archery club, Pedro has to hit a target 3 times. Each time he tries, the probability of hitting is 0.03. By using a suitable approximation, find the probability that it takes him exactly 51 attempts to reach a score of 3 hits.

(Hint: He scores 2 hits in 50 attempts and then a hit at the 51st.)

***6** Cars arrive at a river ferry at random times but at a constant mean rate of 4 cars in each hour. The ferry departs at one-hourly intervals and can carry up to 3 cars at a time. If the ferry is full then any cars not taken on board must wait in a queue until a space becomes available for them on a later ferry.

The 9 a.m. ferry has just departed. No car was left behind when it departed.

(a) Find the probability that

(i) no cars will be left waiting after the 10 a.m. ferry departs

(ii) no cars will be left waiting after the 10 a.m. ferry departs and no cars will be left waiting after the 11 a.m. ferry departs

(iii) just one car will be left waiting after the 10 a.m. ferry departs but no cars will be left waiting after the 11 a.m. ferry departs

(b) Find the overall probability that no cars will be left waiting after the 11 a.m. ferry departs.

***7** Emissions from a radioactive source occur at random times at a mean rate of 0.5 per second. So the number of emissions in one second has a Poisson distribution with mean 0.5.

(a) What is the distribution of the number of emissions in

(i) 2 seconds **(ii)** 3 seconds **(iii)** 4 seconds

The ‘waiting time’ is defined as the time between one emission and the next.

Suppose an emission has just occurred. The waiting time will be less than 1 second if at least one emission occurs in the first second after the emission that has just occurred.

(b) Find the probability that the waiting time is less than 1 second.

The waiting time will be less than 2 seconds if at least one emission occurs in the first 2 seconds after the emission that has just occurred.

(c) Find the probability that the waiting time is less than 2 seconds.

(d) Hence find the probability that the waiting time will be between 1 and 2 seconds.

(e) Find the probability that the waiting time will be between

(i) 2 and 3 seconds **(ii)** 3 and 4 seconds

Emissions from a radioactive source occur at random times at a mean rate of λ per second.

(f) Write down an expression for the mean number of emissions occurring in a period of t seconds.

(g) Show that the probability that the waiting time is between t seconds and $t + h$ seconds is $e^{-\lambda t}(1 - e^{-\lambda h})$.

8 **(a)** The random variable X is such that $X \sim \text{Po}(3)$. Find

(i) $\text{P}(X = 3)$ **(ii)** $\text{P}(X = 2)$

(b) The random variable Y is such that $Y \sim \text{Po}(5)$. Find

(i) $\text{P}(Y = 5)$ **(ii)** $\text{P}(Y = 4)$

(c) Show that if a random variable X is such that $X \sim \text{Po}(\lambda)$, where λ is an integer, then $\text{P}(X = \lambda) = \text{P}(X = \lambda - 1)$.
(Two examples of this are apparent in the graphs on page 22.)

***9** People catch taxis at a certain point in a city. People arrive at the point at random times at a mean rate of one per minute. Taxis arrive independently and at random at a mean rate of one per two minutes. If a person is waiting and a taxi comes, they catch it. Similarly, if a taxi is waiting and a person comes, the taxi picks up the person.

At a particular instant there are no people and no taxis at the point.
Find the probability that one minute later there are no people and no taxis.

Test yourself (answers p 96)

1 The number of misprints on a page of a newspaper follows a Poisson distribution with mean 2.2 per page. Use the Poisson formula to calculate the probability that a particular page contains

(a) no misprints **(b)** exactly one misprint **(c)** two or more misprints

2 Records show that 1.5% of patients treated for a disease are made worse by the treatment. By using a suitable approximation, find the probability that in a random sample of 500 treated patients, 10 or more are made worse.

3 The arrival of visitors to a museum during the middle part of the day can be modelled by a Poisson distribution. If X is the number of visitors arriving per hour, then $X \sim \text{Po}(1.8)$. Find the probability that the number of visitors arriving in a period of 5 hours is

(a) less than 5 **(b)** more than 5

4 There are three entrances to a garden festival. People pass through them at random times at mean rates of 1.0, 1.5 and 2.5 per minute. Find the probability that during a period of 3 minutes a total of 10 people pass through the entrances.

5 **(a)** Write down two conditions needed to be able to approximate the binomial distribution by the Poisson distribution.

A researcher has suggested that 1 in 150 people is likely to catch a particular virus. Assuming that a person catching the virus is independent of any other person catching it,

(b) find the probability that in a random sample of 12 people, exactly 2 of them catch the virus

(c) estimate the probability that in a random sample of 1200 people fewer than 7 catch the virus

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