Mechanics 2 for Edexcel contents

1 Projectiles

In this chapter you will

- learn about the motion of a projectile (a particle moving in two dimensions acted on by gravity)
- solve problems about projectiles, including those involving release at height

Key points from Mechanics 1

- The constant acceleration equations for motion in one dimension are

  $v = u + at$

  $s = \frac{1}{2}(u + v)t$

  $s = ut + \frac{1}{2}at^2$

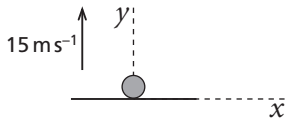
  $s = vt - \frac{1}{2}at^2$

  $v^2 = u^2 + 2as$

- The acceleration due to gravity is approximately equal to $9.8\,\text{m s}^{-2}$.

A Motion of a projectile (answers p 118)

A1 A ball is thrown vertically upwards from ground level with an initial speed of $15\,\text{m s}^{-1}$.



(a) Describe the motion of the ball.

(b) What is the acceleration of the ball?

(c) Use the constant acceleration equations to find

(i) the height of the ball above ground level after 1 second

(ii) the speed of the ball after 1 second

(d) Use the constant acceleration equations to find an expression for

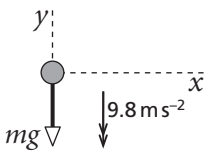
(i) the height of the ball above ground level after t seconds

(ii) the speed of the ball after t seconds

(e) **(i)** What is the speed of the ball when it is at its maximum height?

(ii) Find the maximum height reached by the ball.

A ball is dropped off the edge of a cliff. It starts from rest and moves vertically downwards. The only force acting on the ball is its weight, so the ball has a constant acceleration vertically downwards of magnitude $9.8\,\text{m s}^{-2}$.



A2 **(a)** **(i)** Use the constant acceleration equations to find an expression for the displacement of the ball at time t seconds.

(ii) Complete this table of values for the displacement of the ball.

t	0	1	2	3	4
y					

(b) **(i)** Use the constant acceleration equations to find an expression for the velocity, v_y m s^{-1}, of the ball at time t seconds.

(ii) Complete this table of values for the velocity of the ball.

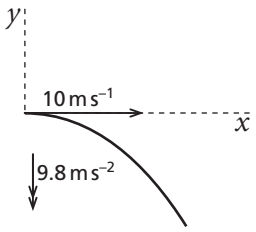
t	0	1	2	3	4
v_y					

An object that is thrown, dropped or launched into the air so that it moves under the influence of gravity alone is known as a **projectile**.
The last two questions involved projectiles that moved vertically in one dimension. We will now extend this into two dimensions.

The projectile will be modelled as a **particle**, an object that has mass but whose size can be ignored. The effects of air resistance and spin are considered negligible. If these effects are not negligible, then there will be forces other than weight acting on the particle, and it will not move as a projectile.

D **A3** Suggest some examples of situations which can be modelled as projectiles. Suggest a situation which cannot be modelled as a projectile and explain why.

Consider the motion of a ball which is kicked off the edge of a cliff so that it moves with an initial horizontal velocity of 10 m s^{-1} as shown.



The horizontal and vertical components of the ball's motion can be considered separately.

There is no force acting on the ball horizontally, so the horizontal component of the velocity remains constant.

The only force acting on the ball is its weight, so the ball has a constant acceleration of 9.8 m s^{-2} vertically downwards and the vertical component of the velocity is not constant.

A4 **(a)** Show that the horizontal component of the displacement, x m, of the ball at time t s is given by $x = 10t$.

(b) By substituting the known values into $s = ut + \frac{1}{2}at^2$, show that the vertical component of the displacement, y m, of the ball at time t s is given by $y = -4.9t^2$.

If **i** and **j** are taken as the unit vectors in the horizontal and vertical directions respectively, then the components of the displacement can be combined and written in vector form.
The displacement, **s**, of the ball at time t is given by $\mathbf{s} = 10t\mathbf{i} - 4.9t^2\mathbf{j}$.

A5 **(a)** Show that $\mathbf{s} = 30\mathbf{i} - 44.1\mathbf{j}$ when $t = 3$.

(b) Complete this table of values for the displacement of the ball.

t	0	1	2	3	4
s				$30\mathbf{i} - 44.1\mathbf{j}$	

(c) Plot these points on graph paper, using a scale of 1 cm to 10 m, and join them to show the path of the ball during the motion. Label the points where $t = 0$, $t = 1$, and so on.

A6 **(a)** Explain why the horizontal component of the velocity, v_x m s^{-1}, of the ball at time t s is given by $v_x = 10$.

(b) By substituting the known values into $v = u + at$, show that the vertical component of the velocity, v_y m s^{-1}, of the ball at time t s is given by $v_y = -9.8t$.

(c) Show that the velocity, **v**, of the ball at time t can be written in vector form as $\mathbf{v} = 10\mathbf{i} - 9.8t\mathbf{j}$.

(d) Complete this table of values for the velocity of the ball.

t	0	1	2	3	4
v	$10\mathbf{i}$				

(e) Using a scale of 1 cm to 10 m s^{-1}, add the velocity vectors to the graph drawn in A5(c). Draw each vector from the point on the graph with the corresponding value of time.

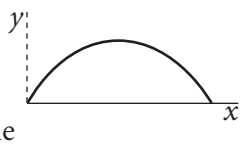
It can be seen from the table of values and the graph that the horizontal component of velocity is constant throughout the motion as there is no force acting on the ball in that direction.
The vertical component of velocity is continuously increasing downwards because of the weight of the ball.

Hence the direction of the velocity of the ball is changing throughout its motion.

D **A7** **(a)** Compare the tables of values for the displacement of the ball found in A2(a) and A5(b). Explain the reasons for the similarities.

(b) Compare the tables of values for the velocity of the ball found in A2(b) and A6(d). Explain the reasons for the similarities.

K Projectile motion takes place in a vertical plane and the path is in the shape of a parabola.



The horizontal component of a projectile's velocity is constant.
The vertical component of a projectile's velocity changes throughout the motion due to its weight.
The magnitude and direction of the velocity of the projectile change throughout the motion.

A8 A ball is thrown with an initial velocity of $(10\mathbf{i} + 15\mathbf{j})\,\mathrm{m\,s^{-1}}$, where **i** and **j** are horizontal and vertical unit vectors respectively. The only force acting on the ball is its weight, so its acceleration is $9.8\,\mathrm{m\,s^{-2}}$ vertically downwards.

(a) (i) By considering the horizontal and vertical components of the motion separately, find an expression in terms of **i** and **j** for the displacement, **s**, of the ball from its starting point at time t.

(ii) Find the displacement of the ball when $t = 0, 0.5, 1, 1.5, 2, 2.5$ and 3.

(iii) Plot these points on graph paper and join them to show the path of the ball during the motion.

(b) (i) By considering the horizontal and vertical components of the motion separately, find an expression in terms of **i** and **j** for the velocity, **v**, of the ball at time t.

(ii) Find the velocity of the ball when $t = 1$, $t = 2$ and $t = 3$.

(iii) Draw the velocity vectors for these times on the graph drawn in (a)(iii) to show how the velocity of the ball changes during the motion.

Example 1

A ball is thrown with an initial velocity of $(4\mathbf{i} + 6\mathbf{j})\,\mathrm{m\,s^{-1}}$.
Find the magnitude and direction of the velocity after 1 second.

Solution

Consider the horizontal and vertical components of motion separately.

Initial horizontal velocity $u_x = 4$, initial vertical velocity $u_y = 6$.

Horizontally the ball is moving with constant velocity. $v_x = u_x = 4$

Vertically the ball is moving with constant acceleration $9.8\,\mathrm{m\,s^{-2}}$ downwards.

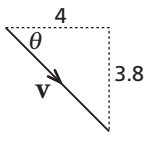
List the known values and the unknowns. $u_y = 6,\ a = -9.8,\ t = 1,\ v_y = ?$

Use $v = u + at$ *to find* v_y. $v_y = 6 - 9.8 \times 1 = -3.8$

The horizontal and vertical components can be combined and written as a vector.

$$\mathbf{v} = 4\mathbf{i} - 3.8\mathbf{j}$$

Sketch the vector.



Use Pythagoras to find the magnitude. $v = \sqrt{4^2 + 3.8^2} = \sqrt{30.44} = 5.52$ to 2 d.p.

Use trigonometry to find the direction. $\tan\theta = \dfrac{3.8}{4} = 0.95 \Rightarrow \theta = 43.5°$ to 1 d.p.

After 1 second the ball is travelling at $5.5\,\mathrm{m\,s^{-1}}$ at 43.5° below the horizontal.

Example 2

A stone is thrown over a cliff with an initial velocity of $12\,\text{m s}^{-1}$ horizontally. The stone hits the water after 5 seconds.
Find the height of the cliff above the water and the speed of the stone as it hits the water.

Solution

Consider the horizontal and vertical components of the motion separately.

Horizontally the stone is moving with constant velocity. $v_x = u_x = 12$

Vertically the stone is moving with constant acceleration $9.8\,\text{m s}^{-2}$ downwards.

Vertically: $u_y = 0,\ a = -9.8,\ t = 5,\ y = ?,\ v_y = ?$

The height of the cliff is the magnitude of y when $t = 5$.

Use $s = ut + \frac{1}{2}at^2$ *to find* y. $y = 0 - \frac{1}{2} \times 9.8 \times 5^2 = -122.5$

The height of the cliff is 122.5 m.

Use $v = u + at$ *to find* v_y. $v_y = 0 - 9.8 \times 5 = -49$

The velocity of the stone when it hits the water is $\mathbf{v} = 12\mathbf{i} - 49\mathbf{j}$.

Find the magnitude of the velocity. $v = \sqrt{12^2 + 49^2} = 50.447\ldots$

The stone hits the water at a speed of $50.4\,\text{m s}^{-1}$ to 1 d.p.

Example 3

A ball is thrown with initial velocity $(8\mathbf{i} + 11\mathbf{j})\,\text{m s}^{-1}$, where **i** and **j** are horizontal and vertical unit vectors.
Find the displacement and velocity of the ball after 2 seconds.

Solution

The only force acting on the ball is its weight, so it is moving as a projectile with an acceleration of $9.8\,\text{m s}^{-2}$ *vertically downwards.*
Consider the horizontal and vertical components of the motion separately.
The initial horizontal component of velocity is $8\,\text{m s}^{-1}$ *and the initial vertical component of velocity is* $11\,\text{m s}^{-1}$.

Consider the horizontal motion. $v_x = u_x = 8$

Find x *when* $t = 2$. $x = 2 \times 8 = 16$

Consider the vertical motion. $u_y = 11,\ a = -9.8,\ t = 2,\ y = ?,\ v_y = ?$

Use $s = ut + \frac{1}{2}at^2$ *to find* y. $y = 11 \times 2 - \frac{1}{2} \times 9.8 \times 2^2 = 2.4$

Use $v = u + at$ *to find* v_y. $v_y = 11 - 9.8 \times 2 = -8.6$

Combine the horizontal and vertical components to find the displacement and velocity vectors.

After 2 seconds the displacement is $(16\mathbf{i} + 2.4\mathbf{j})$ m and the velocity is $(8\mathbf{i} - 8.6\mathbf{j})\,\text{m s}^{-1}$.

Exercise A (answers p 119)

Take **i** and **j** as the horizontal and vertical unit vectors throughout this exercise.

1 An object is projected with an initial velocity of $12\,\text{m s}^{-1}$ horizontally.

(a) Show that its velocity after 2 seconds is $(12\mathbf{i} - 19.6\mathbf{j})\,\text{m s}^{-1}$.

(b) Find its velocity after 4 seconds.

2 An object is projected with initial velocity $(10\mathbf{i} + 20\mathbf{j})\,\text{m s}^{-1}$.

(a) By considering the horizontal and vertical components of the motion separately, find the object's displacement when $t = 0$, 1, 2, 3 and 4.

(b) Plot these displacements on graph paper and join them to show the path of the object.

(c) Find the velocity of the object when $t = 2$. Draw this vector on your graph.

(d) Find the velocity of the object when $t = 4$. Draw this vector on your graph.

3 A ball is thrown with initial velocity $(4\mathbf{i} + 7\mathbf{j})\,\text{m s}^{-1}$.

(a) Find its velocity after 2 seconds.

(b) Find its displacement after 2 seconds.

4 A football is kicked with an initial velocity of $(6\mathbf{i} + 12\mathbf{j})\,\text{m s}^{-1}$.

(a) What is the initial speed of the ball?

(b) What angle does the initial velocity make with the horizontal?

(c) What is the speed of the ball after 2 seconds?

(d) What angle does the velocity make with the horizontal after 2 seconds?

5 A stone is thrown off a cliff with initial velocity $(4\mathbf{i} + 3\mathbf{j})\,\text{m s}^{-1}$. The stone hits the water after 4 seconds.

(a) Find the height of the cliff above the water.

(b) How far from the base of the cliff does the stone hit the water?

6 An arrow is fired from a bow with an initial velocity of $(40\mathbf{i} + 7.5\mathbf{j})\,\text{m s}^{-1}$. It hits the target after 1.5 seconds.

(a) What is the horizontal distance from the bow to the target?

(b) Given that the arrow is fired from a height of 1 metre above ground level, at what height does the arrow hit the target?

***7** A ball is thrown with initial velocity $(5\mathbf{i} + 6\mathbf{j})\,\text{m s}^{-1}$.

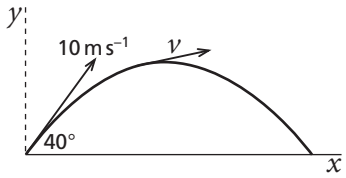
(a) Write an expression for the velocity of the ball at time t.

(b) When is the speed of the ball least?

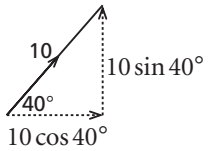
(c) What is the speed at this time?

B Projectile problems (answers p 120)

Consider a particle projected from the origin with velocity $10\,\text{m s}^{-1}$ at an angle of 40° to the horizontal.



The initial velocity can be resolved into horizontal and vertical components; the horizontal and vertical components of the motion can be considered separately.



There is no force acting horizontally, so the horizontal component of the velocity remains constant.

Vertically, the weight of the particle causes an acceleration of $9.8\,\text{m s}^{-2}$ downwards.

At time t, the velocity of the particle can be found using $v = u + at$ in the horizontal and vertical directions.

Horizontally, $v_x = 10\cos 40°$

Vertically, $v_y = 10\sin 40° - 9.8t$

At time t, the displacement of the particle can be found using $s = ut + \frac{1}{2}at^2$ in the horizontal and vertical directions.

Horizontally, $x = 10t\cos 40°$

Vertically, $y = 10t\sin 40° - 4.9t^2$

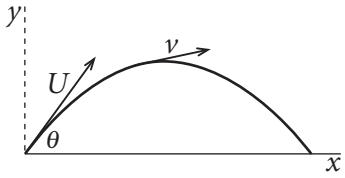
K

At time t, a particle projected from the origin with initial velocity U at an angle of θ to the horizontal will be at the point where

$$x = Ut\cos\theta \quad \text{and} \quad y = Ut\sin\theta - \tfrac{1}{2}gt^2$$

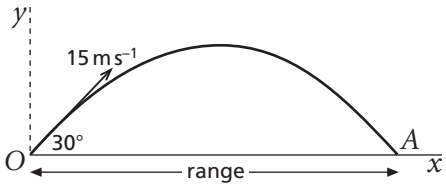
The velocity components are

$$v_x = U\cos\theta \quad \text{and} \quad v_y = U\sin\theta - gt$$



These equations can be used to solve problems involving projectiles.

Consider a particle projected from point O with initial velocity $15\,\text{m s}^{-1}$ at an angle of 30° to the horizontal.
The diagram shows the parabolic path of the projectile.



The **range** of the projectile is the horizontal distance travelled by the projectile from O to A.

The **time of flight** of the projectile is the time taken to travel from O to A, the time that the projectile is in the air.

At points O and A, $y = 0$.

B1 **(a)** Write down the horizontal and vertical components of the initial velocity.

(b) Write an equation for the height of the particle, y, at time t.

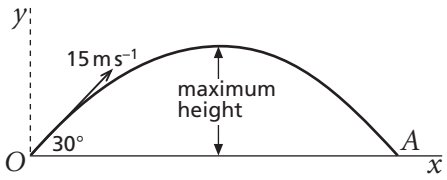
(c) Use this equation to find the time of flight of the projectile.

(d) Write an equation for the horizontal distance x travelled by the particle at time t.

(e) By substituting the value of the time of flight into the equation found in part (d) find the range of the projectile.

The path of the projectile is symmetrical, so the projectile reaches its **maximum height** at the mid-point of the path.

When the projectile is at its maximum height the vertical component of its velocity is zero.



B2 **(a)** Write an equation for the y-component of the velocity of the particle at time t.

(b) Use this equation to find the time when the particle is at its maximum height.

(c) How does this time relate to the time of flight of the particle?

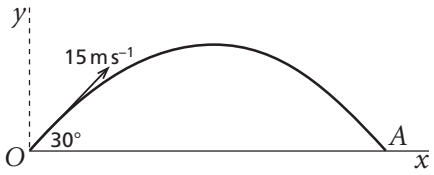
(d) By substituting the time found in part (b) into the equation for the height of the particle, find the maximum height of the projectile.

D

B3 **(a)** What horizontal distance has the particle travelled when it is at its maximum height?

(b) What is the direction of the velocity when the projectile is at its maximum height?

The path of the projectile is in the shape of a parabola. The equation of the path of the projectile can be found by eliminating t from the equations $x = 15t\cos 30°$ and $y = 15t\sin 30° - \frac{1}{2}gt^2$.



B4 **(a)** Rearrange the equation for x to find an equation for t in terms of x.

(b) Substitute the expression for t found in (a) into the equation for y to show that $y = x\tan 30° - \dfrac{gx^2}{450}(1 + \tan^2 30°)$.
(Note that $1 + \tan^2 x = \sec^2 x$.)

(c) Find the height of the particle above ground level when it has travelled 3 metres horizontally.

(d) By forming and solving a quadratic equation find the two possible values of x when the particle is 2 metres above ground level.

B5 A particle is projected from ground level with initial velocity $10\,\text{m s}^{-1}$ at an angle of 20° to the horizontal.

(a) Find the time of flight of the projectile.

(b) Find its range.

(c) Find its maximum height.

Example 4

A particle is projected from ground level with initial velocity $(6\mathbf{i} + 5\mathbf{j})\,\text{m s}^{-1}$.
Find the time of flight and range of the particle.

Solution

Consider the horizontal and vertical components of the motion separately.

Initially $u_x = 6,\ u_y = 5$.

Use $s = ut + \frac{1}{2}at^2$ *for vertical motion.* $y = 5t - \frac{1}{2}gt^2$

The particle hits the ground when $y = 0$. $0 = 5t - 4.9t^2$

$\Rightarrow\ 0 = t(5 - 4.9t)$

$\Rightarrow\ t = 0$ and $1.020\ldots$

Initially $t = 0$, so the time of flight is 1.02 s (to 3 s.f.).

Use $s = ut + \frac{1}{2}at^2$ *for horizontal motion.* $x = 6t + 0 = 6 \times 1.02 = 6.12$

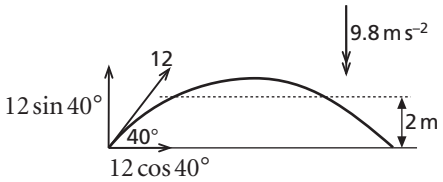
The range of the particle is 6.12 m (to 3 s.f.).

Example 5

A hockey ball is hit from ground level with an initial speed of $12\,\text{m s}^{-1}$ at 40° to the horizontal.
For how long is the ball more than 2 metres above the ground?

Solution

Draw a sketch showing the known values.



Use $s = ut + \frac{1}{2}at^2$ *for vertical motion.* $y = Ut\sin\theta - \frac{1}{2}gt^2$

$2 = 12 \times t \times \sin 40° - \frac{1}{2} \times 9.8 \times t^2$

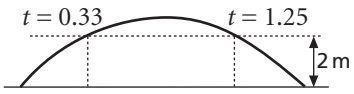
$\Rightarrow\ 2 = 7.713t - 4.9t^2$

Rearrange. $4.9t^2 - 7.713t + 2 = 0$

Solve using the quadratic formula. $t = \dfrac{7.713 \pm \sqrt{7.713^2 - 4 \times 4.9 \times 2}}{2 \times 4.9}$

$\Rightarrow t = 0.33$ or 1.25 to 2 d.p.

These are the two times when the ball is at a height of 2 metres.
The difference between these is the time above that height.



Time above 2 m $= 1.25 - 0.33$

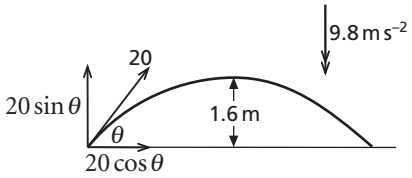
$= 0.92$ s to 2 d.p.

Example 6

A ball is kicked from ground level at a speed of $20\,\text{m s}^{-1}$.
When it is at its maximum height it just passes over a fence 1.6 m high.
Find the angle of projection.

Solution

Draw a sketch showing the known values.



Use $v = u + at$ for vertical motion.

$v_y = U\sin\theta - gt$

At maximum height, $v_y = 0$

so $20\sin\theta - gt = 0$

Rearrange to express t in terms of θ. Leave g in the expression.

$$t = \frac{20\sin\theta}{g}$$

Use $s = ut + \frac{1}{2}at^2$ for vertical motion.

$$y = Ut\sin\theta - \tfrac{1}{2}gt^2$$

$$1.6 = 20\times\frac{20\sin\theta}{g}\times\sin\theta - \tfrac{1}{2}g\times\frac{400\sin^2\theta}{g^2}$$

$$\Rightarrow \quad 1.6 = \frac{200\sin^2\theta}{g}$$

$$\Rightarrow \sin^2\theta = \frac{1.6g}{200}$$

Substitute the value of g at this stage.

$$\sin\theta = \pm\sqrt{\frac{1.6g}{200}} = \pm 0.28$$

We know that θ is less than 90°.

$\theta = 16°$ to the nearest degree

Alternatively you could use $v^2 = u^2 + 2as$ for the vertical motion.

Exercise B (answers p 120)

1 A ball is thrown from ground level with initial speed $20\,\text{m s}^{-1}$ at an angle of 40° to the horizontal.

(a) Find the time of flight of the ball (the length of time the ball is in the air).

(b) Find the range of the ball.

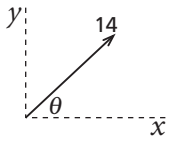
2 A cricket ball is hit from ground level with initial velocity $(4\mathbf{i} + 10\mathbf{j})\,\text{m s}^{-1}$.

(a) Find the time of flight.

(b) Find the range of the cricket ball.

3 A ball is kicked from ground level with initial speed $16\,\text{m s}^{-1}$ at an angle of 25° to the horizontal.
Find the maximum height of the ball.

4 A hockey ball is hit with an initial velocity of $14\,\text{m s}^{-1}$ at an angle θ to the horizontal where $\sin\theta = 0.6$ as shown in the diagram.



(a) Find the maximum height of the ball.

(b) Find the range of the ball.

5 Ann throws a ball to Julian with initial velocity $(7\mathbf{i} + 5\mathbf{j})\,\text{m s}^{-1}$ and Julian catches it at the same height.

(a) For how long is the ball in the air?

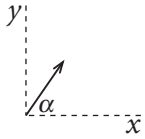
(b) How far apart are Ann and Julian?

6 A particle is projected from the origin with initial speed $V\,\text{m s}^{-1}$ at an angle α to the horizontal.

(a) Show that the time of flight of the particle is given by $t = \dfrac{2V\sin\alpha}{g}$.

(b) Show that the range of the particle is given by $R = \dfrac{2V^2\sin\alpha\cos\alpha}{g}$.

7 A golf ball is hit so that it leaves the ground with an initial velocity of magnitude $15\,\text{m s}^{-1}$, at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$, as shown in the diagram.



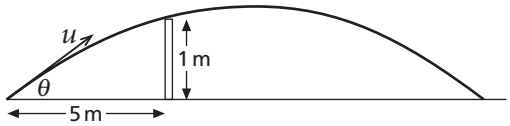
(a) How high does the ball go?

(b) Given that the ground is horizontal, how far does the ball travel before its first bounce?

8 A ball is projected at an initial speed of $15\,\text{m s}^{-1}$ and just passes over a wall 1.8 m high when it is at its maximum height.
Find the angle of projection of the ball.

9 Azmat throws a ball to Susan who is 80 m away and who catches it at the same height as it was thrown from.
The ball is in the air for 5 seconds.
Find the initial speed of the ball and the angle at which it was thrown.

10 A ball is kicked from a point 5 m horizontally away from a wall 1 m high.
The ball just passes over the wall 1 second after it is kicked.



Find the initial speed of the ball and the angle at which it was kicked.

11 An arrow is fired at a target. It hits the target after 2.5 seconds at the same level as it was fired from.
Given that the arrow was fired at an initial speed of $50\,\text{m s}^{-1}$, find the angle of projection and the range of the arrow.

12 Alison kicks a ball with a velocity of $10\,\text{m s}^{-1}$ at an angle α to the horizontal where $\sin\alpha = 0.7$. The ball moves freely under gravity and passes over a wall 2 metres high when it is at its maximum height.

(a) Find the time for the ball to reach its maximum height.

(b) Find the vertical distance between the ball and the top of the wall at this time.

(c) Find the length of time for which the ball is above the height of the wall.

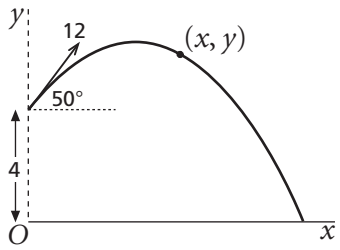
13 A stone is catapulted from ground level at an angle of 35° to the horizontal. Given that the stone hits the ground 25 m from its point of projection, find its speed of release and the time it is in the air.

14 A particle is projected from the origin with initial speed u at an angle θ to the horizontal and moves freely under gravity.
When it has moved a horizontal distance x, its height above the point of projection is y.
Show that $y = x\tan\theta - \dfrac{gx^2}{2u^2}(1 + \tan^2\theta)$.

C Release from a given height

Consider a particle projected with velocity $12\,\text{m s}^{-1}$ at an angle of 50° to the horizontal from a height 4 m above ground level.

At time t, the particle is at the point (x, y).



The horizontal component of the displacement of the particle is unaffected by the height of release.

$$x = 12t\cos 50°$$

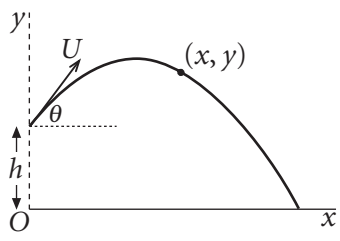
The vertical component of the displacement is 4 m when the motion starts, causing the path of the projectile to be displaced vertically by 4 m.

$$y = 12t\sin 50° - \tfrac{1}{2}gt^2 + 4$$

K

At time t, a projectile released at $U\,\text{m s}^{-1}$ at θ to the horizontal from height h m above the ground is at the point where

$$x = Ut\cos\theta \quad \text{and} \quad y = Ut\sin\theta - \tfrac{1}{2}gt^2 + h$$

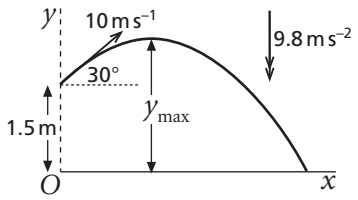


Example 7

A ball is thrown with an initial speed of $10\,\text{m s}^{-1}$ at an angle of 30° to the horizontal from a height of 1.5 m above ground level.
Find its maximum height above the ground and the time when it hits the ground.

Solution

Draw a sketch showing the known values.



Use $v = u + at$ for vertical motion.

$v_y = U\sin\theta - gt$

At the maximum height $v_y = 0$,

so $10\sin 30° - gt = 0$

$$\Rightarrow t = \frac{10\sin 30°}{g}$$

Use $s = ut + \frac{1}{2}at^2$ for vertical motion.

Taking the height of release into account,

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$$y = 10 \times \frac{10\sin 30°}{g} \times \sin 30° - \tfrac{1}{2}g \times \frac{100\sin^2 30°}{g^2} + 1.5$$

$$\Rightarrow y = \frac{50\sin^2 30°}{g} + 1.5 = 2.8 \text{ (to 1 d.p.)}$$

Alternatively you could use $v^2 = u^2 + 2as$ to find the maximum height.

Consider when the ball hits the ground.

The ball hits the ground when $y = 0$,

so $10t\sin 30° - \frac{1}{2} \times 9.8t^2 + 1.5 = 0$

Rearrange.

$4.9t^2 - 5t - 1.5 = 0$

Solve using the quadratic formula.

$$t = \frac{5 \pm \sqrt{5^2 + 4 \times 4.9 \times 1.5}}{2 \times 4.9}$$

$\Rightarrow t = -0.242$ or 1.263

The time is positive, so take the positive root. $t = 1.3$ (to 1 d.p.)

The maximum height of the ball is 2.8 m and it hits the ground after 1.3 s.

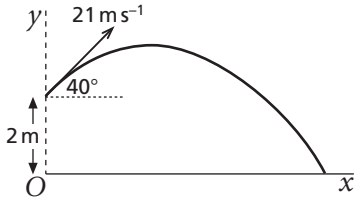
Exercise C (answers p 121)

1 A javelin is thrown from a height of 1.75 m at an angle of 45° to the horizontal with a speed of $25\,\text{m s}^{-1}$.

(a) Find the time when the javelin is at its greatest height.

(b) What is the greatest height of the javelin above the ground?

2 A discus is projected at an angle of 40° with a speed of 21 m s^{-1} and from a height above the ground of 2 m as shown in the diagram.



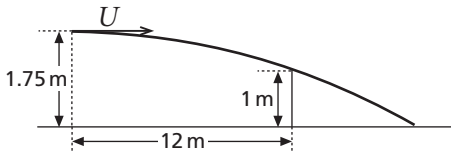
(a) Find an expression for the velocity of the discus at time t.

(b) Find an expression for the position vector with respect to O of the discus at time t.

(c) What is the length of the throw?

3 A coin is thrown into a wishing well at a velocity of $(0.5\mathbf{i} + \mathbf{j})$ m s^{-1} where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and upwards vertical unit vectors.
It is 10 m from the top of the well to the water.
If the coin was thrown from the top of the well, find the time taken for it to hit the water.

4 Tim is standing 12 m away from the net on a tennis court. The net is 1 m high.
He hits the tennis ball horizontally with speed U m s^{-1} from a height of 1.75 m.



(a) Given that the ball just clears the top of the net, find the time taken for the ball to reach the net.

(b) Hence find the value of U.

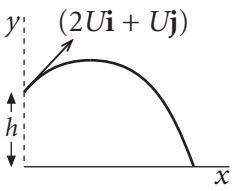
(c) How far beyond the net does the ball land?

5 A small relief plane is flying horizontally at 30 m s^{-1}. Its height is 210 m.
A package, released from the plane, just clears some trees which are 30 m high.

(a) At what horizontal distance from the trees is the package released?

(b) How far beyond the trees does the package land?

6 A particle is projected with initial velocity $(2U\mathbf{i} + U\mathbf{j})$ m s^{-1} from a height h m above ground level.
Find an expression, in terms of U, g and h, for the maximum height reached by the particle.



7 Two children throw stones into the sea.
Jill throws her stone at an angle of 60° to the horizontal and speed 10 m s^{-1}, while Jack throws his stone at a 40° angle and can only manage an initial speed of 5 m s^{-1}.
The stones are thrown simultaneously and both are released at a height 1.4 m above sea level.

(a) What is the maximum height of Jill's stone?

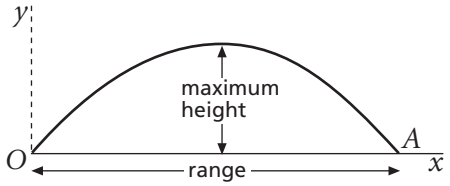
(b) Which stone lands in the water first?

(c) Whose stone lands further away, and how much further is it?

***8** A fairground game involves catapulting an object towards a target on the ground 20 m away.
The object is released at a height of 1 m above ground level.
Assume that the ground is level and that the object moves as a projectile.
Given that the catapult is angled at 45° to the horizontal and the object hits the target, find the speed at which the object leaves the catapult.

Key points

- Projectile motion takes place in a vertical plane in the shape of a parabola. The only force acting is the weight of the particle. (p 8)
- At time t, the horizontal and vertical components of position of a particle projected at an initial velocity $U\,\text{m s}^{-1}$ at an angle of θ to the horizontal are given by
$$x = Ut\cos\theta \quad \text{and} \quad y = Ut\sin\theta - \tfrac{1}{2}gt^2$$ (p 12)
- At time t, the horizontal and vertical components of velocity of a particle projected at an initial velocity $U\,\text{m s}^{-1}$ at an angle of θ to the horizontal are given by
$$v_x = U\cos\theta \quad \text{and} \quad v_y = U\sin\theta - gt$$ (p 12)
- The range of a projectile is the horizontal distance from O to A.
The time of flight is the time taken to travel from O to A.
The maximum height of the projectile occurs at the mid-point of the path of the projectile when the vertical component of the velocity is zero.

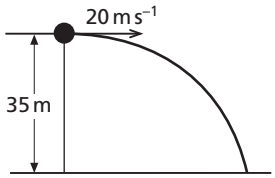


(pp 12–13)
- At time t, a projectile released from height h above the ground is at the point where
$$x = Ut\cos\theta \quad \text{and} \quad y = Ut\sin\theta - \tfrac{1}{2}gt^2 + h$$ (p 17)

Mixed questions (answers p 121)

1 (a) A stone is projected vertically upwards from ground level with initial speed $12\,\text{m s}^{-1}$.
(i) Find the maximum height reached by the stone.
(ii) Find the time the stone is in the air.
(b) The same stone is projected from ground level with initial speed $12\,\text{m s}^{-1}$ at an angle of 60° above the horizontal.
(i) Find the maximum height reached by the stone.
(ii) Find the time the stone is in the air.
(c) Explain why your answers to (a) and (b) are not the same.

2 A ball is hit from the top of a vertical cliff and moves freely under gravity to land in the sea.
The ball is hit from a height of 35 m above the level of the sea and it moves off with an initial velocity of 20 m s^{-1} horizontally as shown in the diagram.



(a) Find the time between the ball being hit and it reaching the sea.

(b) Find the horizontal distance between the foot of the cliff and the point where the stone reaches the sea.

(c) Find the speed of the stone as it reaches the sea.

3 An athlete puts a shot with initial velocity 8 m s^{-1} at an angle of 35° to the horizontal.

(a) Find the range of the shot on horizontal ground, assuming that it is thrown from ground level.

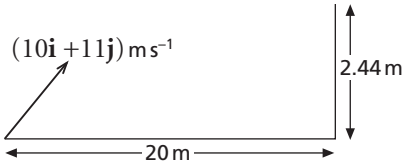
(b) The shot is actually released at a height of 1.5 metres. Find the range of the shot taking into account the height of release.

(c) What modelling assumptions have been made in answering this question?

4 (a) A particle is projected from the origin with initial speed V m s^{-1} at an angle α to the horizontal.
Show that the maximum height of the particle is $\dfrac{V^2 \sin^2 \alpha}{2g}$.

(b) A ball is hit from ground level at an angle α to the horizontal, where $\tan \alpha = \frac{3}{4}$.
The ball reaches a maximum height of 8 m above ground level.
Find the initial speed of the ball.

5 David takes a free kick in a football match.
He kicks the ball with velocity $(10\mathbf{i} + 11\mathbf{j})$ m s^{-1} from a point 20 m from the goal, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors.
The height of the goal is 2.44 m.



(a) How long does the ball take to travel 20 m horizontally?

(b) Assuming that the goalkeeper cannot reach the ball, does David score a goal?

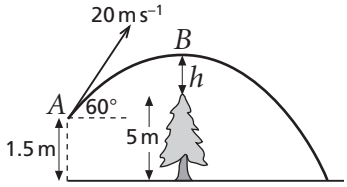
(c) Find the magnitude and direction of the velocity of the ball after it has travelled 20 m horizontally.

6 A particle is projected from point O with initial velocity $(a\mathbf{i} + b\mathbf{j})$ m s^{-1}.
It hits the ground again R m from O after T seconds.

(a) Find a and b in terms of R, T and g.

(b) Find, in terms of T and g, an expression for the maximum height above ground level of the particle.

7 A stone is catapulted from a point A, 1.5 m above ground level, with a speed 20 m s^{-1} at an angle of elevation of 60° as shown in the diagram.
At B, the stone is moving horizontally and it passes over a tree which is 5 m tall.
The motion of the stone can be modelled as that of a particle moving freely under gravity.

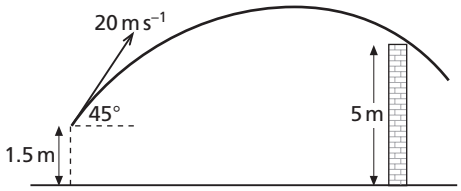


(a) Find the time taken for the stone to reach B.

(b) Find the height, h m, of the stone above the tree at B.

8 A ball is thrown with initial velocity U m s^{-1} so that its range is equal to its maximum height.
What is its angle of projection?

9 A ball is thrown from a height of 1.5 m above horizontal ground with an initial speed of 20 m s^{-1} at an angle of elevation of 45°.
The ball just passes over a vertical wall of height 5 m as shown in the diagram.



(a) Find the maximum height of the ball above the ground.

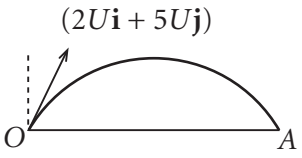
(b) Find the time taken for the ball to reach the wall.

(c) Find the horizontal distance from the initial position of the ball to the wall.

(d) Find the speed of the ball at the instant when it passes over the wall.

(e) Find the direction of motion of the ball at the instant when it passes over the wall.

10 A particle is projected from a point O on a horizontal plane with velocity $(2U\mathbf{i} + 5U\mathbf{j})$.
The particle strikes the plane at A, as shown in the diagram.



(a) Show that the time taken for the particle to travel from O to A is $\frac{10U}{g}$.

(b) Find, in terms of g and U, the distance OA.

(c) Find, in terms of U, the initial speed of the ball.

(d) **(i)** Find, in terms of g and U, the two times during the flight when the particle is moving with speed $\sqrt{5}U$.

(ii) Find the height of the particle above the horizontal plane at these times.

Test yourself (answers p 122)

1 A hockey ball is hit at ground level so that it moves with an initial speed of $18\,\text{m s}^{-1}$ at an angle of 30° above the horizontal. The motion of the ball can be modelled as that of a particle moving freely under gravity.

(a) Find, to three significant figures, the greatest height above the ground reached by the ball.

(b) When the ball has travelled a distance of 25 m horizontally, it hits a wall. Find, to three significant figures, the height above ground at which the ball hits the wall.

(c) State one physical factor that could be taken into account in any refinement of the model which would make it more realistic.

2 A particle P is projected with velocity $(4u\mathbf{i} + 2u\mathbf{j})\,\text{m s}^{-1}$ from a point O on horizontal ground, where $\mathbf{i}$ and $\mathbf{j}$ are horizontal and vertical unit vectors respectively. The particle P hits the ground at point A, a distance of 245 m from O.

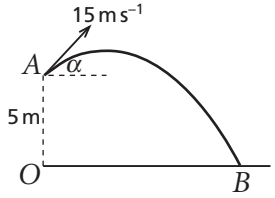
(a) Show that $u = 12.25$.

(b) Find the time of flight from O to A.

(c) Find the speed of the particle at A.

(d) Find the height of the particle above the ground when it has moved a horizontal distance of 140 m.

3 A ball is projected from a point A with an initial speed of $15\,\text{m s}^{-1}$ at an angle α to the horizontal, where $\tan\alpha = \frac{4}{3}$.
The point A is 5 m vertically above O, which is on horizontal ground as shown in the diagram.
The ball hits the ground at point B.

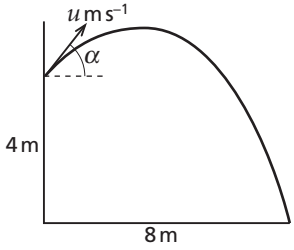


(a) Find the time taken for the ball to move from A to B.

(b) Find the horizontal distance OB.

(c) Find the speed of the ball immediately before it hits the ground.

4 A ball is thrown from a point 4 m above horizontal ground. The ball is projected at an angle α above the horizontal, where $\tan\alpha = \frac{3}{4}$. The ball hits the ground at a point which is a horizontal distance 8 m from its point of projection, as shown. The initial speed of the ball is $u\,\text{m s}^{-1}$ and the time of flight is T seconds.



(a) Prove that $uT = 10$.

(b) Find the value of u.

As the ball hits the ground, its direction of motion makes an angle ϕ with the horizontal.

(c) Find $\tan\phi$.

Edexcel