Core 3 for Edexcel contents

2 Functions

In this chapter you will learn
- what is meant by a function, including one–one and many–one functions
- about the domain and range of a function
- how to find a composite function
- how to find an inverse function and draw its graph

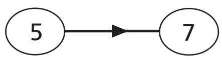
A What is a function? (answers p 160)

The concept of a **function** began its development in the 18th century and is now fundamental to almost every branch of mathematics. A function is essentially a rule or process that generates exactly one output for every given input.

An example of a function is one where the inputs are people and the outputs are their favourite colours. So an input could be Jane Jones and the output would be her favourite colour, say, red.

In mathematics, we often deal with functions where both the inputs and outputs are numbers.

For a rule such as $y = x + 2$, we can think of the values of x as the inputs and the values of y as the outputs. For example, an input of $x = 5$ gives an output of $y = 5 + 2 = 7$.



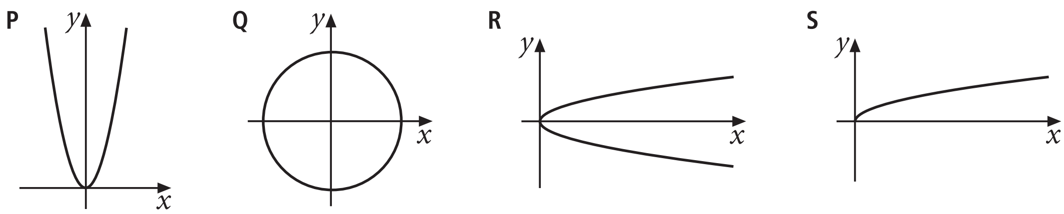
Not all rules generate exactly one output for each given input: not all rules give functions.

A1 Below are four rules that connect x and y.

$y = x^2$	$y^2 = x$	$x^2 + y^2 = 25$	$y = \sqrt{x}$

$\sqrt{x}$ means the **positive** square root of x.

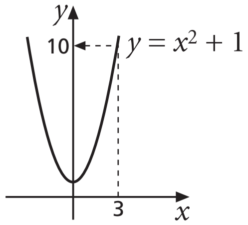
(a) Match each rule with one of the graphs below.



(b) For each rule, find the corresponding value or values of y when $x = 4$.

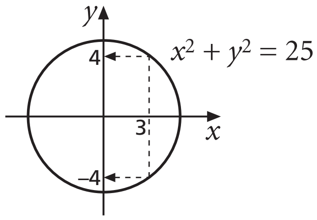
$y = x^2 + 1$ is an example of a rule for a function as each input value of x generates **exactly one** corresponding output value of y.

For example, when $x = 3$ then $y = 10$.



However, $x^2 + y^2 = 25$ is not a rule for a function as each input value of x between –5 and 5 generates **two** corresponding values of y.

For example, when $x = 3$ then $y = 4$ and $y = -4$.



A2 Which of these are not rules for functions?

$y = x^2 - 3$	$y^2 - x^2 = 0$	$y = \sqrt{x + 3}$	$y^3 = x$	$y^4 = x$

A3 For the rule $y = \sqrt{x}$, can you find the value of y when $x = -5$?

When defining a function you need to specify the set of input values to be used. This set of input values is called the **domain** of the function.

For example, the rule $y = \sqrt{x}$ generates one value of y for each input value of x that is positive or 0. We cannot find the square root of a negative number so we can use the rule to define a function if we use non-negative input values. So a suitable domain is the set of values $x \geq 0$.

K

A definition of a function consists of two parts.

The **rule** This tells you how values of the function are calculated.

The **domain** This tells you the set of values to which the rule is applied.

We can use letters to stand for functions, the most usual ones being f, g and h. For example, if we call the square root function f, we can write it as

$$f(x) = \sqrt{x}, \ x \geq 0$$

where the rule is $f(x) = \sqrt{x}$ and the domain is the set of values $x \geq 0$.

(We could write the same function as $f(y) = \sqrt{y}, \ y \geq 0$: using y to stand for the input values does not change the rule or the domain.)

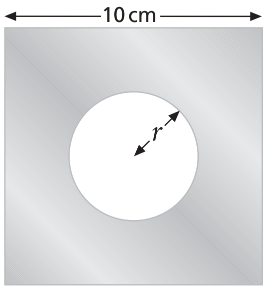
Sometimes, the context in which a rule is applied can restrict the domain.

For example, the rule $g(x) = x^3$ can be applied to any real number x and so the domain of g could be the complete set of real numbers (the complete set of rational and irrational numbers).

However, if we use the rule $g(x) = x^3$ to determine the volume of a cube of side x, it would be inappropriate to include negative values in the domain.

A4 A circle is cut out of a square of metal as shown.
The centre of the circle is at the centre of the square.

The radius of the circle is r.
The area of metal remaining is A(r).



(a) Which of the following is the rule for the function A?

$A(r) = 10 - \pi r^2$	$A(r) = \pi r^2 - 100$
$A(r) = 100 - \pi r^2$	$A(r) = \pi r^2 - 10$

(b) Find the value of A(3), correct to two decimal places.

(c) Which of the following do you think is an appropriate domain for the function A?

$r \leq 100$	$r \leq 5$	$0 \leq r \leq 10$	$0 \leq r \leq 100$	$0 \leq r \leq 5$	$r \leq 10$

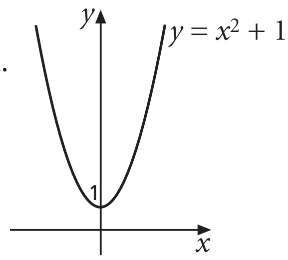
To sketch a graph of a function, you need to exclude input values that are not in its domain.

Example 1

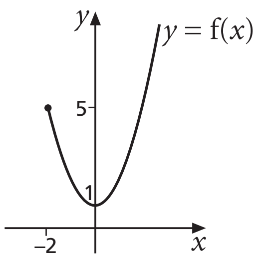
Sketch the graph of $y = f(x)$ where $f(x) = x^2 + 1$, $x \geq -2$.

Solution

The graph of the rule $y = x^2 + 1$ for all real values of x is as shown.



The domain of f is $x \geq -2$.
$f(-2) = (-2)^2 + 1 = 5$ so
the graph of $y = f(x)$ is as shown.



The point (–2, 5) is included in the graph, so is shown by a solid circle.

An open circle can be used to show that an end-point is not included in a graph.

A useful shorthand for the set of real numbers is $\mathbb{R}$.

The statement '$x \in \mathbb{R}$' means 'x belongs to the set of real numbers'.

Sometimes we want the domain of a function to be all real numbers except a particular value.

For example, $f(t) = \dfrac{1}{t-2}$ is not defined for $t = 2$ (as division by 0 is not possible)
so a suitable domain for f is $t \in \mathbb{R}, t \neq 2$ (the set of all real numbers except 2).

A5 Sketch the graph of $y = \mathrm{f}(x)$ for each function below.

(a) $\mathrm{f}(x) = x^2 - 1,\ x \le 3$ **(b)** $\mathrm{f}(x) = 2^x,\ x \in \mathbb{R}$

(c) $\mathrm{f}(x) = 3x + 1,\ x > -1$ **(d)** $\mathrm{f}(x) = \dfrac{2}{x},\ x \in \mathbb{R},\ x \ne 0$

There is an alternative notation for the rule of a function.
For example, rather than writing $\mathrm{f}(x) = x^2$ we can write

$\mathrm{f}: x \mapsto x^2$ (Read '$\mathrm{f}: x \mapsto x^2$' as 'f, such that x maps to x^2')

This notation suggests the idea of the input x being 'converted' to the output x^2.
You can also use an ordinary arrow ($\rightarrow$) for this purpose.

For example, the input 3 is converted to the input 3^2 or 9.
We could write either $\mathrm{f}(3) = 9$ or $\mathrm{f}: 3 \mapsto 9$.

A6 Sketch the graph of $y = \mathrm{f}(x)$ where $\mathrm{f}: x \mapsto \sqrt{x+2},\ x \ge -2$.

Exercise A (answers p 160)

1 Match each function to its sketch graph.

(a) $\mathrm{f}(x) = x^2,\ x \ge 1$ **(b)** $\mathrm{f}(x) = x^2,\ x \ge -2$ **(c)** $\mathrm{f}(x) = x + 2,\ x \in \mathbb{R}$

(d) $\mathrm{f}(x) = x + 2,\ x \le 0$ **(e)** $\mathrm{f}(x) = \sqrt{x},\ x \ge 1$ **(f)** $\mathrm{f}(x) = \sqrt{x-1},\ x \ge 1$

(g) $\mathrm{f}(x) = \dfrac{1}{x},\ x \in \mathbb{R},\ x \ne 0$ **(h)** $\mathrm{f}(x) = \dfrac{1}{x},\ x > 1$

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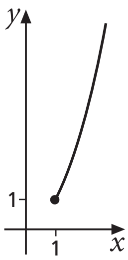

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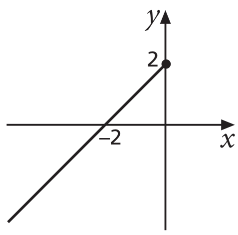

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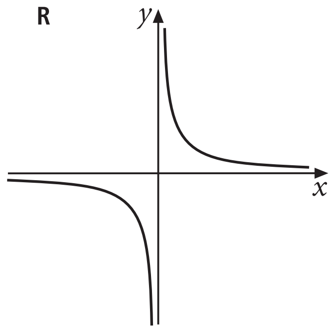

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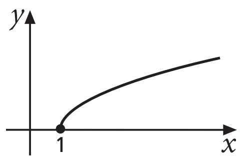

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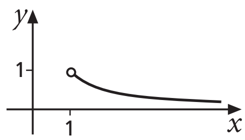

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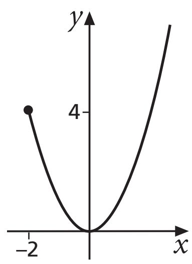

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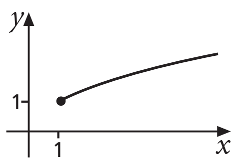

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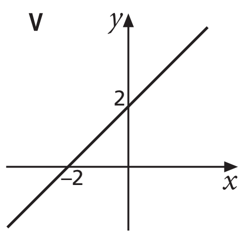


2 Sketch the graph of $y = f(x)$ for each of these functions.

(a) $f(x) = x - 2,\ x \le 1$ **(b)** $f(x) = x^2 - 3,\ x > -2$

(c) $f: x \mapsto 9 - x^2,\ x \in \mathbb{R}$ **(d)** $f: x \mapsto \sqrt{x} + 2,\ x \ge 0$

(e) $f(x) = x - 3,\ -1 \le x \le 10$ **(f)** $f(x) = x^2,\ -2 < x < 4$

3 A rule for a function is $f(c) = \dfrac{1}{c+3}$.

(a) Evaluate **(i)** $f(2)$ **(ii)** $f(-2)$ **(iii)** $f(0)$ **(iv)** $f(-5)$

(b) Which value of c cannot be included in the domain of f?

B Many–one and one–one functions (answers p 161)

B1 **(a)** For the function $f(x) = 2x + 7,\ x \in \mathbb{R}$, solve the equation $f(x) = 25$.

(b) For the function $g(x) = x^2,\ x \in \mathbb{R}$, solve the equation $g(x) = 4$.

(c) For the function $h(x) = x^3,\ x \in \mathbb{R}$, solve the equation $h(x) = 27$.

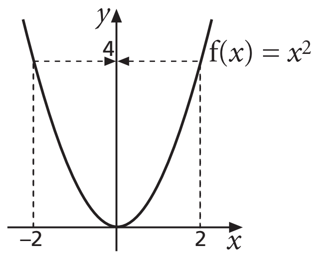
K A **many–one function** is a function where there are two or more different inputs that generate the same output.

For example, for the function $f(x) = x^2$

we have $f(2) = 2^2 = 4$

and $f(-2) = (-2)^2 = 4$

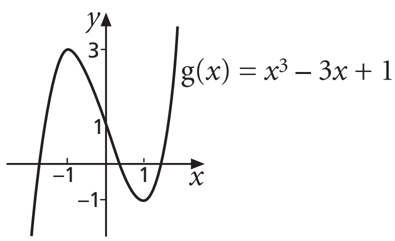
Two different inputs (–2 and 2) generate the same output (4) so the function $f(x) = x^2$ is many–one.



K Any function which is not many–one is a **one–one function**. Each output can be generated by only one input.

B2 A function g is defined for all real values of x by $g(x) = x^3 - 3x + 1$. A sketch of $y = g(x)$ is shown.

Explain how the sketch shows that g is many–one.



B3 A function is defined by $f(x) = x^3 - x,\ x \in \mathbb{R}$.

(a) Evaluate f(1), f(0) and f(–1).

(b) Is f a one–one function?

B4 Classify each function as one–one or many–one.

(a) $f(x) = x^2 - 2,\ x \in \mathbb{R}$ **(b)** $f(n) = n^3,\ n \in \mathbb{R}$

(c) $f: t \mapsto 3t + 5,\ t \in \mathbb{R}$ **(d)** $f: x \mapsto x^4,\ x \in \mathbb{R}$

Exercise B (answers p 161)

1 Two functions are defined as

$$f(x) = x^2 + 1, \ x < 1$$
$$g(x) = x^2 + 1, \ x > 1$$

(a) Draw a sketch graph for each function.

(b) Which one of these functions is one–one?

2 A function is defined by $f(t) = t^2 + 2t, \ t \geq -2$.

(a) Solve the equation $f(t) = 0$.

(b) Show that the equation $f(t) = 3$ has only one solution.

3 **(a)** Evaluate $\sin 0$ and $\sin \pi$.

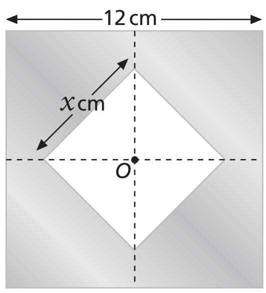
(b) Show that the function $g(\theta) = \sin\theta$ with domain $0 \leq \theta \leq 2\pi$ is many–one.

(c) Solve the inequality $g(\theta) > \frac{1}{2}$.

4 A function is defined by $h: x \mapsto \cos x, \ 0 \leq x \leq \pi$.
Is the function h one–one or many–one? Explain how you decided.

5 A square is cut out of a square piece of metal as shown.
The point O is the centre of both squares.

The length of one edge of the smaller square is x cm.
The area of metal remaining is $A(x)$ cm^2.



(a) Show that the rule for the function A is

$$A(x) = 144 - x^2$$

(b) What is a suitable domain for the function A?

(c) **(i)** Sketch the graph of $y = A(x)$.

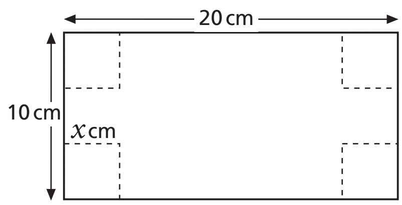
(ii) Classify the function A as one–one or many–one.

(d) Solve the equation $A(x) = 100$.

***6** A manufacturer has some sheets of card.
Each sheet measures 20 cm by 10 cm.

A cuboid-shaped box is to be made from each sheet by cutting a square (x cm by x cm) from each corner and folding up.

The sheet of card is assumed to be of negligible thickness and rigid.



(a) Show that the volume of the box, $V(x)$ cm^3, is given by

$$V(x) = 4x^3 - 60x^2 + 200x$$

(b) What is a suitable domain for the function V?

(c) Show that the function V is many–one.

(d) The manufacturer wants to make boxes with a volume of 144 cm^3.
What value of x should the manufacturer use?

C The range of a function

K The **range** of a function is the complete set of possible output values.

For example, consider the function f defined by $f(x) = x^2 - 4x + 5,\ x > 0$.

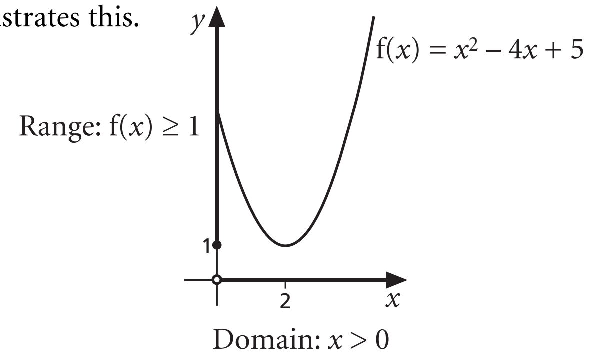
In completed-square form $x^2 - 4x + 5 = (x-2)^2 - 4 + 5$

$= (x-2)^2 + 1$

which is a minimum when $x = 2$.

The value $x = 2$ is in the domain of f so the minimum value of f(x) is f(2) which is 1.
Hence the range is the set of all the real numbers greater than or equal to 1.
We can write the range as $f(x) \geq 1$.

A sketch graph illustrates this.



Example 2

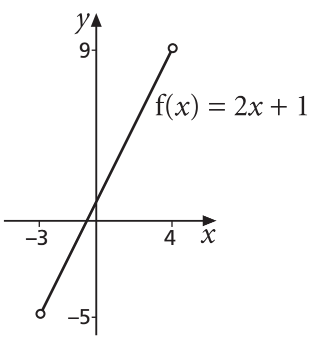
Draw a sketch graph of the function defined by $f(x) = 2x + 1,\ -3 < x < 4$.
State its range.

Solution

The graph is a straight line so find the end-points.

When $x = -3$, $2x + 1 = 2\times-3 + 1 = -5$

When $x = 4$, $2x + 1 = 2\times4 + 1 = 9$



The end-points (–3, –5) *and* (4, 9) *are not included in the graph and so are shown by open circles.*

The range is $-5 < f(x) < 9$.

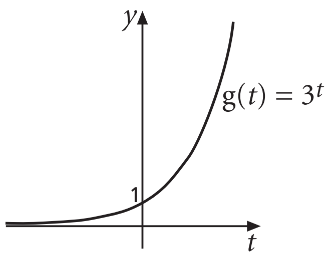
Example 3

Draw a sketch graph of the function defined by $g(t) = 3^t,\ t \in \mathbb{R}$.
State its range.

Solution

$g(0) = 3^0 = 1$

As t becomes large and positive, $g(t)$ gets larger very quickly.
As t becomes large and negative, $g(t)$ gets smaller and smaller, getting closer and closer to 0.



The graph gets closer and closer to the horizontal axis but does not reach it, so the range is $g(t) > 0$.

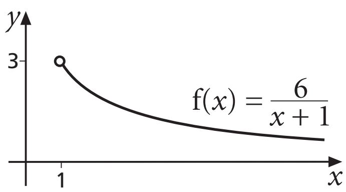
Example 4

Draw a sketch graph of the function defined by $f(x) = \dfrac{6}{x+1},\ x > 1$.
State its range.

Solution

When $x = 1$, $\dfrac{6}{x+1} = \frac{6}{2} = 3$ so $(1, 3)$ is the left-hand end point.
As x becomes large and positive, $f(x)$ gets smaller and smaller, getting closer and closer to 0.

A sketch is



The graph gets closer and closer to the horizontal axis but does not reach it so the range is $0 < f(x) < 3$.

Exercise C (answers p 162)

1 For each function below, draw a sketch graph and state the range of the function.

(a) $f(x) = x + 3,\ x \geq 1$

(b) $g(x) = x^2 + 3,\ x \in \mathbb{R}$

(c) $h(x) = 1 - x^2,\ x \in \mathbb{R}$

(d) $f(x) = \sin x,\ 0 \leq x \leq 2\pi$

(e) $g(x) = 3^x,\ x > -1$

(f) $h(x) = \dfrac{4}{x-1},\ x \geq 2$

(g) $f(x) = (x - 2)^2 + 3,\ x \in \mathbb{R}$

(h) $g(x) = (x + 1)^2 - 2,\ x \geq 0$

2 **(a)** Write the expression $x^2 - 6x + 10$ in completed-square form.

(b) **(i)** Sketch the graph of $y = f(x)$ where f is defined by

$$f: x \mapsto x^2 - 6x + 10, \ x \leq 2$$

(ii) Find the range of f.

3 **(a)** Sketch the graph of the function $g(\theta) = \cos(2\theta)$ where the domain of g is $0 \leq \theta \leq \frac{1}{2}\pi$.

(b) State the range of g.

4 The function f is defined by $f(x) = x^2 + 2x + 6, \ x \in \mathbb{R}$.

(a) Find the range of f.

(b) Hence show that the equation $f(x) = 3$ has no real solution.

5 The function h is defined as $h(t) = t^2 - 4t - 5, \ -1 \leq t \leq 6$.

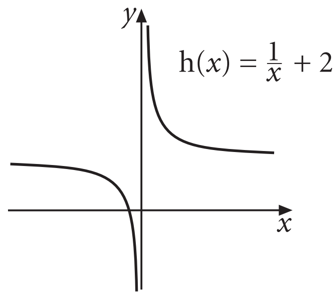
(a) Show that this function is many–one.

(b) Find the range of h.

6 The function h is defined by

$$h(x) = \frac{1}{x} + 2, \ x \in \mathbb{R}, \ x \neq 0$$

A sketch of its graph is shown.



(a) Solve these equations.

(i) $h(x) = 2\frac{1}{3}$ **(ii)** $h(x) = 3$

(iii) $h(x) = 4$ **(iv)** $h(x) = 1$

(b) Explain why the equation $h(x) = 2$ has no solution.

(c) Which set of values below is the range of h?

$h(x) \in \mathbb{R}$	$h(x) \in \mathbb{R}, \ h(x) \neq 2$	$h(x) \in \mathbb{R}, \ h(x) \neq 0$

7 What is the range of each function?

(a) $f(x) = \frac{1}{x} - 5, \ x \in \mathbb{R}, \ x \neq 0$ **(b)** $f(x) = \frac{4}{x} + 3, \ x \in \mathbb{R}, \ x \neq 0$

8 The function g is defined by

$$g(x) = \frac{1}{x}, \ x \in \mathbb{R}, \ x \geq 2$$

Sketch the graph of the function and find its range.

9 The function f is defined by

$$f: t \mapsto 2^{-t}, \ t > -2$$

Find the range of f.

10 The function g is defined by

$$g(x) = \frac{1}{\sqrt{x}}, \quad x \geq \tfrac{1}{9}$$

(a) Solve the equation $g(x) = 2$.

(b) Find the range of g.

11 The function h is defined by

$$h(x) = 3 + \frac{4}{x-1}, \quad x < 1$$

Find the range of h.

D Composite functions (answers p 163)

D1 A gas meter indicates the amount of gas in cubic feet used by a consumer.
The number of therms of heat from x cubic feet of gas is given by the function f where

$$f(x) = 1.034x, \quad x \geq 0$$

A particular gas company's charge in £ for t therms is given by the function g where

$$g(t) = 15 + 0.4t, \quad t \geq 0$$

(a) How many therms of heat are produced from 500 cubic feet of gas?

(b) What is the cost of using 100 therms?

(c) Find the cost of using these amounts of gas from this gas company.

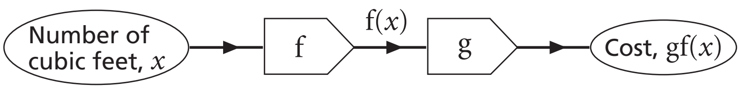
(i) 100 cubic feet **(ii)** 400 cubic feet **(iii)** 1000 cubic feet

To find the cost of using, say, 50 cubic feet of gas you must use both functions: first f and then g. This can be illustrated by the following diagram.

We can write this in stages as $gf(50) = g(f(50)) = g(51.7) = 28.18$.

D2 Evaluate gf(200).

For an input of x cubic feet of gas we have the following diagram.



The function gf is called a **composite** function as it is a composition of two functions, f and g. gf means first f and then g.

D3 **(a)** Find the rule, in its simplest form, for the function gf where $gf(x)$ is the cost of using x cubic feet of gas.

(b) Use your rule to find the cost of 550 cubic feet of gas.

Example 5

Functions f and g are defined for all real values of x by $f(x) = x^2 + 1$
and $g(x) = 5x - 7$

Calculate fg(2), gf(2) and ff(2).

Solution

$fg(2) = f(g(2)) = f(5\times 2 - 7) = f(3) = 3^2 + 1 = 10$

$gf(2) = g(f(2)) = g(2^2 + 1) = g(5) = 5\times 5 - 7 = 18$

$ff(2) = f(f(2)) = f(2^2 + 1) = f(5) = 5^2 + 1 = 26$

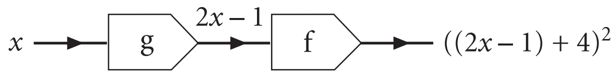
Example 6

Functions f and g are defined for all real values of x by $f(x) = (x + 4)^2$
and $g(x) = 2x - 1$

Find an expression for fg(x).

Solution

$$\begin{aligned} fg(x) = f(g(x)) &= f(2x - 1) \\ &= ((2x - 1) + 4)^2 \\ &= (2x + 3)^2 \end{aligned}$$



Example 7

Functions g and h are defined by $g(x) = \dfrac{6}{x},\ x \in \mathbb{R},\ x \neq 0$

and $h(x) = \dfrac{2}{x + 5},\ x \in \mathbb{R},\ x \neq -5$

Find and simplify an expression for gh(x).

Solution

$$\begin{aligned} gh(x) = g(h(x)) = g\left(\frac{2}{x+5}\right) &= \frac{6}{\left(\frac{2}{x+5}\right)} \\ &= 6 \times \frac{x+5}{2} \\ &= 3(x+5) \end{aligned}$$

D4 Functions f and g are defined for all real values of x by

$f(x) = 10 - x^2$

and $g(x) = 3x + 2$

(a) Evaluate these.

(i) fg(0) **(ii)** gf(4) **(iii)** ff(–2)

(b) Find an expression for each of these.

(i) gf(x) **(ii)** fg(x) **(iii)** gg(x)

Exercise D (answers p 163)

1 Functions f and g are defined for all real values of x by

$f(x) = x^2$

and $g(x) = 3x + 1$

(a) Evaluate these.

(i) $fg(2)$ **(ii)** $gf(2)$ **(iii)** $gg(2)$

(b) Find an expression for $gf(x)$.

(c) Show that $fg(x) = 9x^2 + 6x + 1$.

(d) **(i)** Find an expression, in its simplest form, for $gg(x)$.

(ii) Use your result to evaluate $gg(-1)$.

(iii) Solve the equation $gg(x) = 49$.

2 For each pair of rules below, find expressions for $fg(x)$ and $gf(x)$.

(a) $f(x) = 2x + 3$, $g(x) = x^3$ **(b)** $f(x) = x^2$, $g(x) = \frac{1}{x+1}$ **(c)** $f(x) = 3x + 2$, $g(x) = 5 - x$ **(d)** $f(x) = 1 - x^2$, $g(x) = 1 - 2x$

3 Functions f and g are defined for all real values of x by

$f: x \mapsto x^2 - 4x + 1$

$g: x \mapsto kx + 5$, where k is a constant

Given that $gf(1) = 2$, find the value of k.

4 Functions f and g are defined by

$f(x) = \frac{1}{x}, \; x > 0$

$g(x) = \frac{1}{3x-1}, \; x > 1$

Find an expression for $fg(x)$ and write down the domain of fg.

5 Functions g and h are defined by

$g: x \mapsto 2 - \frac{6}{x}, \; x > 0$

$h: x \mapsto x + 3, \; x > 0$

(a) Show that $gh(x) = \frac{2x}{x+3}$.

(b) Solve $gh(x) = 1$.

6 Functions f and g are defined by

$f: x \mapsto x - 10, \; x \in \mathbb{R}$

$g: x \mapsto \sqrt{x}, \; x \geq 0$

(a) Find $f(1)$.

(b) Explain why the composite function gf cannot be formed.

7 Functions f and g are defined by

$$f(x) = \frac{1}{(x-1)(x+3)}, \ x > 1$$

$$g(x) = \frac{5}{x}, \ x > 0$$

(a) Solve $gf(x) = 25$.

(b) Write down the domain of gf.

8 A theatre manager notices that if he raises the temperature on the central heating thermostat he can increase the sales of ice cream in the interval.

He observes that the proportion of the audience buying ices is given by the function P where

$$P(c) = 1 - \frac{10}{c}, \ 15 \le c \le 25$$

and where c is the temperature in degrees Celsius.

(a) What proportion of the audience buys ice cream when the temperature is 15 °C?

(b) At what temperature will half of the audience buy ices?

(c) What is the range of the function P?

The function $f(t) = \frac{5}{9}(t - 32)$ gives the temperature in degrees Celsius where t is the temperature in degrees Fahrenheit.

(d) What proportion of the audience buys ice cream when the temperature is 65 °F?

(e) Find the rule for the composite function that determines the proportion of the audience that will buy ices when the temperature is t °F.

(f) One evening, 55% of the audience buy ices.
What is the temperature in degrees Fahrenheit?

***(g)** Work out an appropriate domain for the function Pf.

E Inverse functions (answers p 163)

E1 Functions f and g are defined for all real values of x by

$$f(x) = 2x + 5$$

$$g(x) = \frac{x-5}{2}$$

(a) Evaluate gf(5) and fg(5).

(b) Evaluate gf(–3) and fg(–3).

(c) Find an expression for gf(x).

K

A function that reverses, or 'undoes', the effect of f is its **inverse** and is denoted by f^{-1}.
So $f^{-1}f(x) = x$. It is also true that $ff^{-1}(x) = x$.
(In this context, f^{-1} does not mean $\frac{1}{f}$.)

So we can write $f(x) = 2x + 5$ and $f^{-1}(x) = \frac{x-5}{2}$.

E2 The rule for the function f is $f(x) = 3x - 1$.

(a) (i) Show that $f(7) = 20$.

(ii) Hence write down the value of $f^{-1}(20)$.

(b) Which of the rules below is the rule for the inverse of f?

A $f^{-1}(x) = 3x + 1$

B $f^{-1}(x) = \frac{1}{3}x + 1$

C $f^{-1}(x) = \frac{x+1}{3}$

D $f^{-1}(x) = \frac{x-1}{3}$

K One way to find a rule for the inverse of a function is to write the function in terms of x and y and then rearrange to obtain a rule for x in terms of y.

For example, to find the rule for the inverse of $g(x) = 2x - 1$
first write the rule as $y = 2x - 1$
then rearrange to obtain $y + 1 = 2x$

$$\Rightarrow \tfrac{1}{2}(y + 1) = x$$

So we can write the inverse as $g^{-1}(y) = \frac{1}{2}(y + 1)$

We could write the inverse rule as, say, $g^{-1}(k) = \frac{1}{2}(k + 1)$ or $g^{-1}(p) = \frac{1}{2}(p + 1)$ or use any other letter we choose.

We usually use x which gives $g^{-1}(x) = \frac{1}{2}(x + 1)$ for the inverse rule.

E3 For each rule, find an expression for $f^{-1}(x)$, where f^{-1} is the inverse of f.

(a) $f(x) = 4x + 3$ **(b)** $f(x) = \frac{1}{5}x + 3$ **(c)** $f: x \mapsto 2(x - 7)$

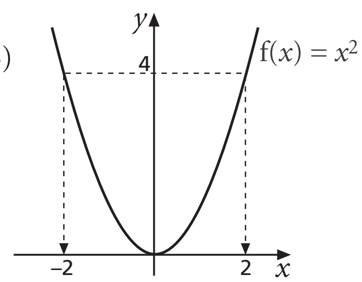
E4 For each rule, find an expression for $g^{-1}(x)$, where g^{-1} is the inverse of g.

(a) $g(x) = 10 - x$ **(b)** $g: x \mapsto -x$ **(c)** $g(x) = \frac{1}{x}$

K An inverse exists for any one–one function.
If a function is many–one, then an inverse function does not exist.

For example, for the many–one function $f(x) = x^2$ we have two different inputs (–2 and 2) that generate the same output (4) so reversing the effect of f gives two possible outputs for an input of 4.

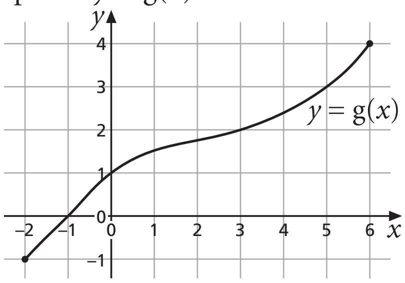
A function must give exactly one output for each input so an inverse function for $f(x) = x^2$ does not exist.



E5 Which of these functions does not have an inverse?

A $f(x) = x^2 - 5,\ -3 \le x \le 3$ **B** $f(x) = x^2 - 5,\ 1 \le x \le 6$

E6 The domain of a function g is $-2 \le x \le 6$. The graph of $y = g(x)$ is shown.



(a) Write down the value of

(i) $g(3)$ **(ii)** $g^{-1}(3)$ **(iii)** $g^{-1}(4)$ **(iv)** $g^{-1}(-1)$

(b) Why is it not possible to evaluate $g^{-1}(10)$?

(c) What is the domain of the function g^{-1}?

(d) Write down the range of the function g^{-1}.

(e) Sketch the graph of the function $y = g^{-1}(x)$.

K The domain of an inverse function f^{-1} is the range of the function f.

E7 For each function below,

(i) using the same scale on the x- and y-axes, sketch the graph of $y = f(x)$

(ii) find an expression for $f^{-1}(x)$

(iii) find the domain of f^{-1}

(iv) add the graph of $y = f^{-1}(x)$ to your sketch of $y = f(x)$

(a) $f(x) = 2x + 1,\ x \ge -2$ **(b)** $f(x) = \frac{1}{4}(x - 2),\ x < 10$

(c) $f(x) = x^2,\ x > 1$ **(d)** $f: x \mapsto x^3 + 1,\ 0 \le x \le 2$

K Using the same scale on the x- and y-axes, the graphs of a function and its inverse have reflection symmetry in the line $y = x$.

Example 8

A function f is defined for $x \ge 0$ by $f(x) = (x + 1)^2 + 2$.
Find an expression for the inverse $f^{-1}(x)$.
Sketch the graph of the inverse function f^{-1} and state its domain.

Solution

First write the function in terms of x and y. $y = (x + 1)^2 + 2$

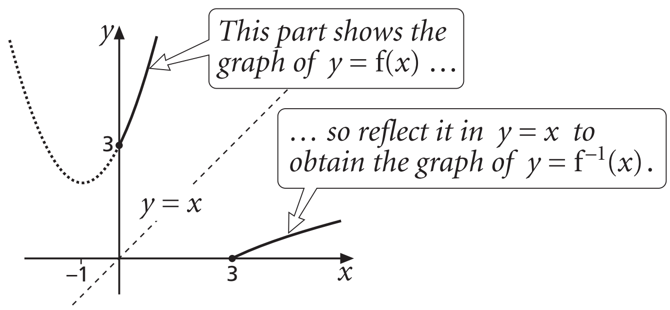
$\Rightarrow \quad y - 2 = (x + 1)^2$

Take the positive square root as $x \ge 0$. $\sqrt{y - 2} = x + 1$

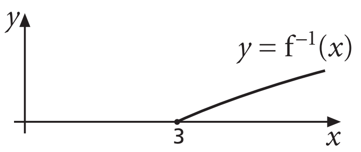
$\Rightarrow \quad \sqrt{y - 2} - 1 = x$

We need an expression for $f^{-1}(x)$. So $f^{-1}(x) = \sqrt{x - 2} - 1$

You can first sketch the graph of $y = (x + 1)^2 + 2$ for $x \in \mathbb{R}$, remembering to use the same scale on the x- and y-axes.



The graph is



The range of f is $f(x) \geq 3$ so the domain of f^{-1} is $x \geq 3$.

Example 9

A function g has the rule defined by $g: x \mapsto \dfrac{3}{2x-1}, \ x \geq 1$

Find an expression for the inverse $g^{-1}(x)$ and find the domain of g^{-1}.

Solution

First write the rule in terms of x and y. $\quad y = \dfrac{3}{2x-1}$

Multiply both sides by $(2x - 1)$. $\quad y(2x-1) = 3$

Expand the brackets. $\quad 2xy - y = 3$

$\Rightarrow \quad 2xy = 3 + y$

$\Rightarrow \quad x = \dfrac{3+y}{2y}$

We need an expression for $g^{-1}(x)$. $\quad$ So $g^{-1}(x) = \dfrac{3+x}{2x}$

The domain of g^{-1} *is the range of* g.

When $x = 1$, $g(x) = \dfrac{3}{2x-1} = 3$ and when $x > 1$, $\dfrac{3}{2x-1} < 3$.

As x becomes large and positive, $\dfrac{3}{2x-1}$ gets smaller and smaller, getting closer and closer to 0.

So the range of g is $0 < g(x) \leq 3$.

Hence, the domain of g^{-1} is $0 < x \leq 3$.

Exercise E (answers p 164)

1 Function f is defined by $f(x) = 5x + 1, \ x \in \mathbb{R}$.
Find an expression for $f^{-1}(x)$.

2 Function g is defined by $g(x) = \frac{1}{4}x - 3, \ 0 \leq x \leq 16$.

(a) Find an expression for $g^{-1}(x)$.

(b) Find the domain and range of g^{-1}.

3 The function $f(t) = \frac{5}{9}(t - 32)$ gives the temperature in degrees Celsius where t is the temperature in degrees Fahrenheit. The domain of the function is $t \geq -459.4$.

(a) Find an expression for $f^{-1}(t)$.

(b) What is the domain of f^{-1}?

(c) Convert –70 °C into a temperature in degrees Fahrenheit.

4 Explain why no inverse exists for function f where f is defined by $f(x) = x^2, \ x \geq -4$.

5 Function h is defined by $h: x \mapsto 2 - 3x,\ x > 0$.

(a) Find an expression for $h^{-1}(x)$.

(b) What is the domain of h^{-1}?

(c) Solve the equation $h^{-1}(x) = h(x)$.

6 Function f is defined by $f(x) = (x - 2)^2 - 5,\ x > 2$.

(a) Find an expression for $f^{-1}(x)$.

(b) What is the domain of f^{-1}?

7 Function g is defined by $g: x \mapsto x^2 + 6x + 10,\ x > -2$.

(a) Write $x^2 + 6x + 10$ in completed-square form.

(b) Find an expression for $g^{-1}(x)$.

8 Find an expression for each inverse function $f^{-1}(x)$ and write down its domain.

(a) $f(x) = \sqrt{x} + 2,\ x \geq 0$

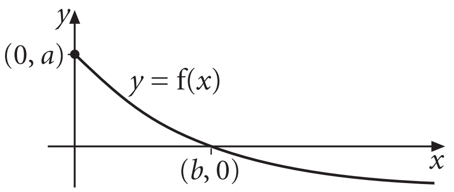
(b) $f(x) = x^3 - 5,\ x \in \mathbb{R}$

(c) $f(x) = \dfrac{2}{3 + x},\ x > 0$

(d) $f(x) = \dfrac{5}{x} - 4,\ x < 0$

9 The graph sketched is of the function defined by

$$f: x \mapsto 3^{-x} - \tfrac{1}{3},\ x \geq 0$$



(a) Work out the values of a and b.

(b) Sketch the curve with equation $y = f^{-1}(x)$.

(c) What is the domain of f^{-1}?

(d) What is the range of f^{-1}?

10 Function f is defined by $f(x) = \dfrac{2x + 3}{x - 2}$.

Show that $f^{-1}(x) = f(x)$ for $x \in \mathbb{R},\ x \neq 2$.

11 Function g is defined as $g: x \mapsto x^2 - 2x + 5,\ x > 1$.

(a) Find an expression for $g^{-1}(x)$.

(b) Sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$.

(c) State the domain and range of g^{-1}.

(d) Solve the equation $g(x) = 7$, leaving your solution in surd form.

(e) Show that the equation $g^{-1}(x) = g(x)$ has no real solutions.

Key points

- A **function** is a rule or process that generates exactly one output for every input. A definition of a function consists of
  a **rule** that tells you how values of the function are calculated
  a **domain** that is the set of values to which the rule is applied (p 21)
- An alternative notation for the rule of a function is to use an arrow. For example, rather than writing $f(x) = x^2$ we can write $f: x \mapsto x^2$. (p 23)
- A **many–one function** is a function where two or more different inputs generate the same output. Any function where each output can be generated by only one input is a **one–one function**. (p 24)
- The **range** of a function is the complete set of possible output values. (p 26)
- The function gf is called a **composite** function and tells you to 'do f first and then g': $gf(x) = g(f(x))$. (p 29)
- A function that reverses, or 'undoes', the effect of f is its **inverse** and is denoted by f^{-1}. So $f^{-1}f(x) = x$.
  Only one–one functions have inverses.
  The domain of an inverse function f^{-1} is the range of the function f. (pp 32–34)
- One way to find a rule for the inverse of a function is to write the function in terms of x and y and then rearrange to obtain a rule for x in terms of y. (p 33)
- Using the same scale on the x- and y-axes, the graphs of a function and its inverse have reflection symmetry in the line $y = x$. (p 34)

Mixed questions (answers p 165)

1 The function f is defined for all real values of x by $f(x) = (x + 2)(x - 2)$.

(a) Sketch the graph of $y = f(x)$, showing where the graph crosses both axes.

(b) Find the range of f.

(c) Explain why the function f does not have an inverse.

2 The function f has domain $x \geq 0$ and is defined by $f(x) = (x + 1)^2 - 3$.

(a) Find the range of f.

(b) Explain why the equation $f(x) = -3$ has no solution.

(c) **(i)** Write down the domain of f^{-1} where f^{-1} is the inverse of f.

(ii) Find an expression for $f^{-1}(x)$.

3 The function f is defined by

$$f: t \mapsto 5^t, \quad t \leq 2$$

Find the range of f.

4 $f(x) = \dfrac{2}{x-1} - \dfrac{6}{(x-1)(2x+1)}$, $x > 1$.

(a) Prove that $f(x) = \dfrac{4}{2x+1}$.

(b) Find the range of f.

(c) Find $f^{-1}(x)$.

(d) Find the range of f^{-1}. Edexcel

5 Functions f and g are defined by

$f: x \mapsto \dfrac{x}{x-3}$, $x \in \mathbb{R}$, $x \neq 3$

$g: x \mapsto \dfrac{1}{2x-1}$, $x \in \mathbb{R}$, $x \neq \frac{1}{2}$

(a) **(i)** Show that $gf(x) = 1 - \dfrac{6}{x+3}$.

(ii) Solve $gf(x) = 7$.

(b) **(i)** Find an expression for $f^{-1}(x)$.

(ii) Find the domain of f^{-1}.

6 Function k is defined by $k(x) = \sqrt{x-1} + 3$, $x \geq 1$.

(a) Find an expression for $k^{-1}(x)$.

(b) Explain why the x-coordinate of any point of intersection of the graphs of $y = k(x)$ and $y = k^{-1}(x)$ must satisfy the equation $k^{-1}(x) = x$.
Hence, find the x-coordinate of any point of intersection.

Test yourself (answers p 166)

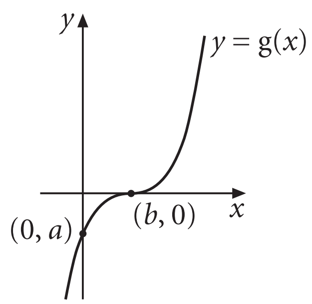
1 Functions f and g are defined for all real values of x by

$f: x \mapsto 1 + 3x$

$g: x \mapsto (x-1)^3$

(a) The composite function gf is defined for all real values of x.
Find $gf(x)$, expressing your answer in its simplest form.

(b) The sketch shows the graph of $y = g(x)$.



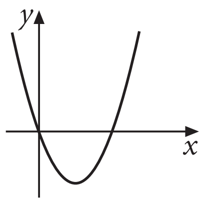
(i) What are the values of a and b?

(ii) Copy the graph of $y = g(x)$ and on the same axes sketch the graph of $y = g^{-1}(x)$.

(c) Find an expression for $g^{-1}(x)$.

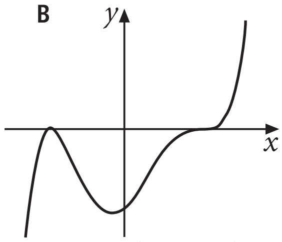
2 One of the following sketch graphs does not represent a function.
State which one this is and give a reason for your answer.

A



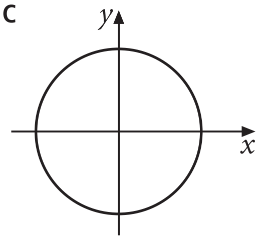
$y = (x-1)^2 - 1$

B



$y = (x-1)^3(x+1)^2$

C



$y^2 + x^2 = 1$

3 Function f is defined by $f(x) = 2x^2,\ x \geq 0$.

(a) What is the domain of $f^{-1}(x)$?

(b) The graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$ intersect at two points.
Find the x-coordinates of these two points.

4 The function h is defined for all real numbers by $h(x) = 8 - 3x^2$.

(a) Find the range of h.

(b) Solve the equation $h(x) = 5$.

(c) Explain why the function h does not have an inverse.

5 Functions f and g are defined by

$$f: x \mapsto \frac{10}{3+x},\quad x > 0$$

$$g: x \mapsto \frac{5}{x},\quad x \in \mathbb{R},\quad x \neq 0$$

(a) **(i)** Find gf(x), giving your answer in its simplest form.

(ii) What is the domain of gf?

(b) What is the range of f?

(c) $f^{-1}(x)$ can be written in the form $\frac{A}{x} + B$ where A and B are constants.

(i) Find the values of A and B.

(ii) Solve the equation $f^{-1}(x) = f(x)$.

6 Functions f and g are defined for all real values of x by

$$f(x) = x^2 - 4x + 7$$

$$g(x) = x + k, \text{ where } k \text{ is a positive constant}$$

(a) Find the range of f.

(b) Given that $fg(3) = 12$, find the value of k.

(c) Solve the equation $gf(x) = 14$.