Core 4 for AQA contents

3 Parametric equations

In this chapter you will learn how to
- work with curves defined by two parametric equations, including the circle and ellipse
- convert between parametric and cartesian equations

A Coordinates in terms of a third variable (answers p 149)

Computer animators make objects on the screen change their position over time. An object's position at any moment can be given using (x, y) coordinates (also known as 'cartesian coordinates'). To instruct the computer to produce a required movement, the x-coordinate and y-coordinate can be separately defined in terms of time.

A1 A computer animator uses these equations to define the movement of a dot on the screen (x and y are in centimetres; t is in seconds).

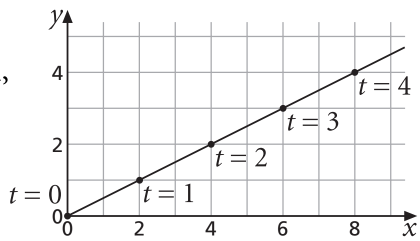
$$x = 3t$$
$$y = 6t - t^2$$

(a) Copy this table and use the equations to complete it.

t	0	1	2	3	4	5	6
x		3					
y		5					

(b) On squared paper, using axes labelled x and y, plot the motion of the dot.
What might the dot represent?

A2 Here a dot has been made to move in a straight line.



(a) By reading off values of x at $t = 1$, $t = 2$ and so on, state an equation for x in terms of t.

(b) Similarly, express y as a function of t.

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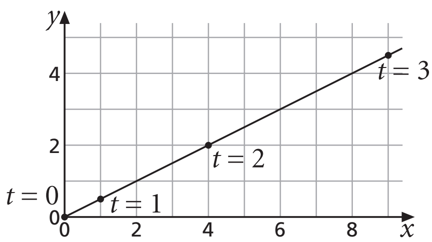
A3 Here, too, a dot moves in a straight line.



(a) How does the motion differ from that in question A2?

(b) Give an equation for x as a function of t

(c) Give an equation for y as a function of t.

K

Two equations that separately define the x- and y-coordinates of a graph in terms of a third variable are called **parametric equations**.

The third variable is called the **parameter**.

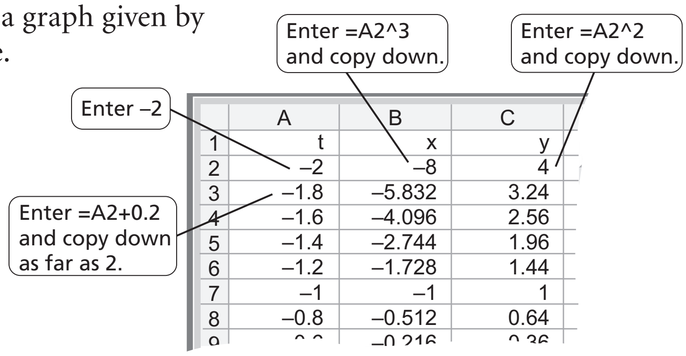
If you have done Mechanics 1, you have already seen the x- and y-coordinates of a projectile's position separately defined as functions of time.

We have used t for the parameter here, but the parameter does not have to represent time, and a different letter could be used.

Many graph plotting calculators and programs will let you define a graph parametrically. The manual (often available on-line) should tell you how. You may be restricted to using t as the parameter.

You can also use a spreadsheet to plot a graph given by a pair of parametric equations, as here.

Column A contains values of t, increasing in steps of 0.2.
Column B gives the x-coordinate, defined by the function $x = t^3$.
Column C gives the y-coordinate, defined by the function $y = t^2$.



To plot the graph, first select the whole of columns B and C.
On the chart toolbar, select the 'scatter chart' button.
(In some versions of Excel, you have to make a 'First column contains …' selection; if so choose 'category (x)-axis labels' or 'x-values for xy-chart'.)

This method plots unjoined points for the graph. You may be able to obtain a scatter diagram with lines drawn between the points, but using a joined-up line chart option does not work with parametric equations on some spreadsheet programs.

A4 Obtain the graph given by $x = t^3,\ y = t^2\ (-2 \le t \le 2)$ by plotting on squared paper or by using a graph plotter or spreadsheet. Call this graph G.

A5 Each of the following graphs is obtained by applying a transformation to graph G. In each case, plot the graph and describe the transformation.

(a) $x = t^3,\ y = t^2 + 2$ **(b)** $x = t^3 - 1,\ y = t^2$ **(c)** $x = 3t^3,\ y = t^2$

(d) $x = t^3,\ y = \frac{1}{2}t^2$ **(e)** $x = t^3,\ y = -t^2$ **(f)** $x = t^3 + 4,\ y = t^2 - 1$

An advantage of defining a graph parametrically is that transformations are simple to apply.

K

To apply a translation $\begin{bmatrix} a \\ b \end{bmatrix}$ add a to the function for x and b to the function for y.

To stretch in the x-direction, multiply the x-function by the required factor; similarly, to stretch in the y-direction, multiply the y-function.

To reflect in the y-axis multiply the x-function by –1; to reflect in the x-axis multiply the y-function by –1.

A6 State a pair of parametric equations for each of these.

(a) The graph $x = t^2 + t,\ y = 1 - t$ translated by $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(b) The graph $x = t^2 + t,\ y = 1 - t$ stretched by a factor of 2 in the x-direction

You can find where a parametrically defined graph meets an axis, or a line parallel to an axis, as follows.

Example 1

Find the coordinates of the points where the curve $x = t^3 + 4$, $y = t^2 - t$ meets the line $y = 12$.

Solution

Substitute 12 for y in the y-equation. $12 = t^2 - t$

$\Rightarrow \quad t^2 - t - 12 = 0$

Factorise. $(t - 4)(t + 3) = 0$

$\Rightarrow \quad t = 4 \text{ or } -3$

Substitute these values of t into the x-equation. When $t = 4$, $x = 4^3 + 4 = 68$
When $t = -3$, $x = (-3)^3 + 4 = -23$

So the points are (68, 12) and (–23, 12).

Exercise A (answers p 149)

1 A curve is defined by $x = \frac{1}{t}$, $y = t^2$.
Find the coordinates of the points on the curve where $t = -3, -2, -1, 1, 2$ and 3.

2 A curve K is defined by $x = t$, $y = t^2$.
Give the parametric equations of a curve obtained by

(a) stretching K by factor 3 in the y-direction

(b) translating K by the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

3 For each of the pairs of parametric equations below,

(i) find the cartesian coordinates for $t = -2, -1, 0, 1, 2$, then plot these points and sketch the graph

(ii) obtain the graph using a graph plotter or spreadsheet; if it differs from your sketch try to sort out why this has happened

(a) $x = t + 4$, $y = 1 - t^2$ **(b)** $x = 2 - t$, $y = t^3 - 2t$ **(c)** $x = t^3$, $y = t^2 - t$

4 Use the fact that $x = 0$ for all points on the y-axis to find where the curve defined by $x = t^2 - 4$, $y = t^3 + t$ meets the y-axis.

5 Find the coordinates of the point(s) where each of the following curves meets the y-axis.

(a) $x = t - 2$, $y = 2t + 1$ **(b)** $x = 3 - t$, $y = t^2 - t$

(c) $x = t^2 + t - 2$, $y = 3t - 6$ **(d)** $x = 8 - t^3$, $y = \frac{t-1}{t+1}$

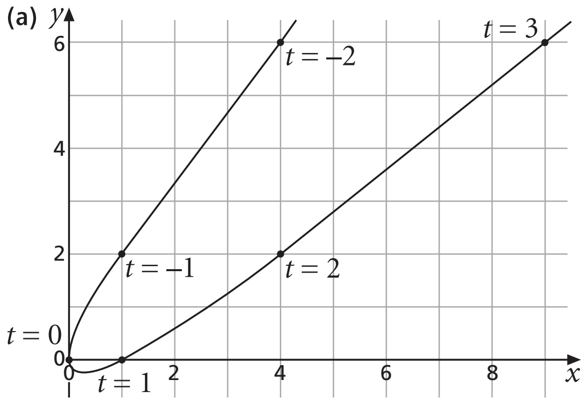
6 Find the coordinates of the points where the curves in question 5 meet the x-axis.

7 Show that the curve $x = \sqrt{t+1},\ y = \dfrac{t}{t^2 - 2}$ meets the line $y = 1$ at (0, 1) and ($\sqrt{3}$, 1).

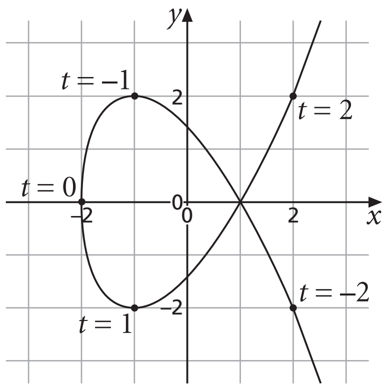
8 The curve $x = t^2 + 1,\ y = t^3 - 1$ meets the line $x = 3$ at points *A* and *B*. Find the exact length of the line segment *AB*.

***9** On the graphs below, the value of the parameter *t* is shown at each dot. The parametric equations involve only low powers of *t*. Write down possible parametric equations and test them on a graph plotter or spreadsheet.

(a)



(b)



B Converting between parametric and cartesian equations (answers p 150)

B1 A graph is defined by these parametric equations.

$x = 2t - 1$
$y = 4t$

(a) Copy this table and complete it using the equations.

t	–2	–1.5	–1	–0.5	0	0.5	1	1.5	2
x									
y									

(b) Make *t* the subject of the second equation.

(c) Substitute this expression for *t* into the first equation, and show that this leads to

$2x - y + 2 = 0$

(d) Check that this equation is consistent with the last two lines of the table.

(e) What type of graph is this?

B2 Use the method of parts (b) and (c) in the previous question to convert each pair of parametric equations to an equation of the form $ax + by + c = 0$.

(a) $x = 3t + 4,\ y = 2t$ **(b)** $x = 5 - 2t,\ y = \frac{1}{3}t$

B3 Here again are the parametric equations from question A1:

$$x = 3t, \; y = 6t - t^2$$

(a) Make t the subject of the first equation.

(b) Show that substituting this expression for t into the second equation gives

$$y = 2x - \frac{x^2}{9}$$

(c) Check that the last two lines in your table from question A1 are consistent with this formula.

The method of questions B1–B3 can often be used to convert a pair of parametric equations into the equation that connects x and y directly (the cartesian equation). It involves first choosing the simpler equation and making t the subject of it.

B4 Obtain a single cartesian equation for each pair of parametric equations. Your equation does not have to be of the form $y = \ldots$

(a) $x = 5t, \; y = 4t - 3$ **(b)** $x = t^2, \; y = 2t$ **(c)** $x = 2t, \; y = t^3$

Example 2

Obtain a cartesian equation for the graph defined by $x = t^3, \; y = \sqrt{t} \quad (t \geq 0)$.

Solution

The second equation is the simpler.	$y = \sqrt{t}$
Square both sides.	$y^2 = t$
Substitute for t in the first equation.	$x = (y^2)^3$
	$\Rightarrow \; x = y^6$

Example 3

Obtain a cartesian equation for the graph defined by $x = t + 5, \; y = t^3 - t$.

Solution

The first equation is the simpler.	$x = t + 5$
Make t the subject.	$t = x - 5$
Substitute for t in the second equation.	$y = (x - 5)^3 - (x - 5)$
Expand the brackets.	$y = x^3 - 15x^2 + 75x - 125 - x + 5$
Simplify.	$y = x^3 - 15x^2 + 74x - 120$

B5 Obtain a single cartesian equation for each pair of parametric equations.

(a) $x = \sqrt{t}, \; y = t^2$ **(b)** $x = t + 3, \; y = 3 - t^2$ **(c)** $x = t + 1, \; y = t^3 - 3t$

The examples opposite show a variety of ways of eliminating the parameter. In example 4, it is t^2, rather than t, that is first made the subject of one of the equations and eliminated. Example 5 again involves making t the subject, though the working is harder. In example 6, the method is one that happens to work with the given equations.

Example 4

Obtain a cartesian equation for the graph defined by $x = 3t^2 - 4,\ y = 8 - t^2$.

Solution

Take the y-equation and make t^2 the subject. $t^2 = 8 - y$

Substitute $8 - y$ for t^2 in the x-equation. $x = 3(8 - y) - 4$

Tidy the equation. $x = 20 - 3y$

Example 5

Obtain a cartesian equation for the graph defined by $x = \dfrac{t}{t+1},\ y = \dfrac{t}{t-3}\ (t \neq -1, t \neq 3)$.

Solution

Multiply both sides of the x-equation by t + 1. $tx + x = t$

Bring all the t-terms together. $tx - t = -x$

$$\Rightarrow \quad t(x-1) = -x$$

$$\Rightarrow \quad t = \frac{-x}{x-1}$$

You will also need an expression for t – 3.

$$t - 3 = \frac{-x}{x-1} - 3$$

$$= \frac{-x}{x-1} - \frac{3(x-1)}{x-1}$$

$$= \frac{-x-3(x-1)}{x-1}$$

$$= \frac{-4x+3}{x-1}$$

Substitute the expressions for t and t – 3 into the y-equation.

$$y = \frac{-x}{x-1} \div \frac{-4x+3}{x-1} = \frac{-x}{x-1} \times \frac{x-1}{-4x+3} = \frac{-x}{-4x+3}$$

So the required cartesian equation is $y = \dfrac{-x}{-4x+3}$.

Example 6

Obtain a cartesian equation for the graph defined by $x = t + \dfrac{1}{t},\ y = t - \dfrac{1}{t}\ (t \neq 0)$.

Solution

Add the two parametric equations. $x + y = 2t$

Subtract the second parametric equation from the first. $x - y = \dfrac{2}{t}$

Multiply the 'sum' equation by the 'difference' equation. $(x + y)(x - y) = 4$, or $x^2 - y^2 = 4$

A statement like $t \neq -2$ or $t \geq 0$ after a pair of parametric equations is often warning you about a value or values of the parameter for which one or more of the equations is invalid (perhaps because division by zero or finding the square root of a negative number would be involved). You do not normally have to do anything about this.

Exercise B (answers p 150)

1 Obtain a single cartesian equation for each pair of parametric equations.

(a) $x = \frac{1}{4}t,\ y = 5t - 1$

(b) $x = 4t,\ y = \dfrac{4}{t}\ \ (t \neq 0)$

(c) $x = t^2 + 4t,\ y = \frac{1}{3}t$

(d) $x = \dfrac{1}{t},\ y = \frac{1}{2}t\ \ (t \neq 0)$

(e) $x = 2 - t,\ y = t^2 + 4$

(f) $x = 2t - 1,\ y = 3 - 4t$

(g) $x = \sqrt{t},\ y = t^2 + t\ \ (t \geq 0)$

(h) $x = \dfrac{1}{t},\ y = 4 - t\ \ (t \neq 0)$

(i) $x = t^3 - t,\ y = t + 2$

(j) $x = 1 + t,\ y = 2t^2 - t^3$

(k) $x = \dfrac{t-1}{2},\ y = (t+1)(t+2)$

(l) $x = 4t^3 + 3,\ y = 6 - t^3$

(m) $x = \sqrt{t},\ y = t(5 - t)\ \ (t \geq 0)$

(n) $x = \dfrac{1}{t-3},\ y = t^2\ \ (t \neq 3)$

(o) $x = \dfrac{1}{t-2},\ y = t^3\ \ (t \neq 2)$

(p) $x = \dfrac{1}{t+2},\ y = \dfrac{1}{t-1}\ \ (t \neq -2, t \neq 1)$

(q) $x = 5 - \sqrt{t},\ y = 4\sqrt{t} - 3\ \ (t \geq 0)$

(r) $x = \dfrac{t}{2t-1},\ y = \dfrac{t}{t-1}\ \ (t \neq \frac{1}{2}, t \neq 1)$

2 A curve is defined parametrically by $x = t^3 + \dfrac{3}{t},\ y = t^3 - \dfrac{3}{t}\ \ (t \neq 0)$.
By first expressing $x + y$ and $x - y$ in terms of t, show that $(x + y)(x - y)^3 = 432$.

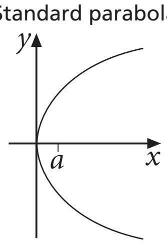
3 A curve is defined by $x = \dfrac{1}{t},\ y = \dfrac{1}{t(t-1)}$. By first expressing $x + y$ and $\dfrac{y}{x}$ in terms of t, find the cartesian equation.

***4** Show that the graph defined by $x = \dfrac{1}{2t-1},\ y = \dfrac{t}{2t-1}$ is a straight line.

Useful curves

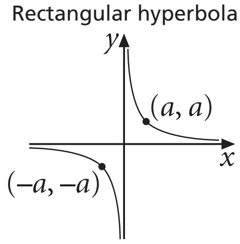
Parametric equations can be used to study curves with interesting mathematical properties that have uses in optics and other branches of science and technology.
Here are two. In each case a is a scale factor. Try working out their cartesian equations.

Standard parabola



$x = at^2,\ y = 2at$

Rectangular hyperbola



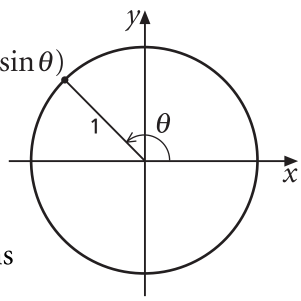
$x = at,\ y = \dfrac{a}{t}$

‘Rectanglar’ means the asymptotes are at right angles.

C Circle and ellipse (answers p 151)

You know from your work on trigonometry that $\cos\theta$ and $\sin\theta$ are defined as the x- and y-coordinates of a point rotating around a circle of radius 1 unit:

$$x = \cos\theta, \; y = \sin\theta$$



You can also regard these equations as defining x and y in terms of a third value θ. Seen this way, they are a pair of parametric equations for a circle, with θ as the parameter.

C1 Put $x = \cos\theta, \; y = \sin\theta$ into a graph plotter and check that you get a circle with its centre at the origin.

C2 What do you expect these pairs of parametric equations to give? Check with a graph plotter.

(a) $x = 3\cos\theta, \; y = 3\sin\theta$ **(b)** $x = 0.5\cos\theta, \; y = 0.5\sin\theta$

In section A we applied a translation to a parametrically defined graph by adding a constant to the x-equation or the y-equation (or both).

C3 Write parametric equations for each of these, then check on a graph plotter.

(a) A circle with unit radius, centre (0, 2)

(b) A circle with unit radius, centre (–3, 0)

(c) A circle with unit radius, centre (1, –6)

(d) A circle with radius 2 units, centre (5, 4)

K The curve $x = r\cos\theta, \; y = r\sin\theta$ is a circle with radius r, centre the origin.
The curve $x = r\cos\theta + p, \; y = r\sin\theta + q$ is a circle with radius r, centre (p, q).

C4 Sketch these circles, indicating the radius and the position of the centre in each case.

(a) $x = \cos\theta - 3, \; y = \sin\theta + 2$ **(b)** $x = \cos\theta + 1, \; y = \sin\theta$

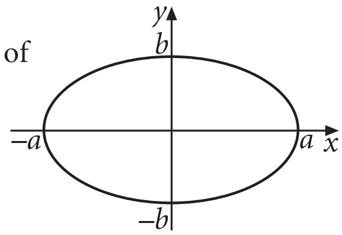
(c) $x = 2\cos\theta - 5, \; y = 2\sin\theta + 5$ **(d)** $x = 0.6\cos\theta, \; y = 0.6\sin\theta - 3$

C5 Put each of these pairs of equations into a graph plotter.
What transformation of the circle $x = \cos\theta, \; y = \sin\theta$ do you obtain in each case?

(a) $x = \cos\theta, \; y = 2\sin\theta$ **(b)** $x = 4\cos\theta, \; y = \sin\theta$

(c) $x = \cos\theta, \; y = 0.6\sin\theta$ **(d)** $x = 0.5\cos\theta, \; y = 1.2\sin\theta$

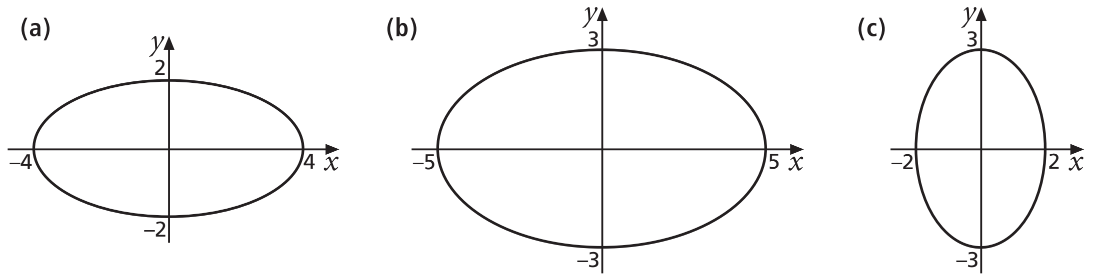
When you stretch a circle with unit radius, centre the origin, by factors of a in the x-direction and b in the y-direction you get an **ellipse** that cuts the x-axis at $(-a, 0)$ and $(a, 0)$ and cuts the y-axis at $(0, b)$ and $(0, -b)$.



K

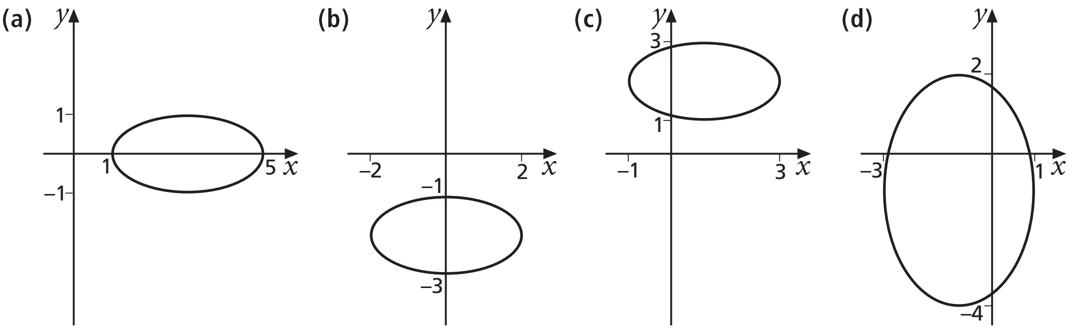
The curve $x = a\cos\theta,\ y = b\sin\theta$ is an ellipse, centre the origin.
Its width is $2a$ units and its height is $2b$ units.

C6 Write a pair of parametric equations for each of these ellipses.



Like a circle, an ellipse can also be defined parametrically when its centre is not the origin.

C7 Write a pair of parametric equations for each of these ellipses.



To convert a pair of parametric equations for an ellipse to a cartesian equation, first obtain expressions for $\cos\theta$ and $\sin\theta$ from the parametric equations, then use the identity $\cos^2\theta + \sin^2\theta = 1$, as in the next example.

Example 7

An ellipse is defined by the parametric equations $x = 4\cos\theta,\ y = 5\sin\theta$.
Find its cartesian equation.

Solution

From the x-equation, $\cos\theta = \dfrac{x}{4}$ From the y-equation, $\sin\theta = \dfrac{y}{5}$

Substituting these expressions in $\cos^2\theta + \sin^2\theta = 1$, $\left(\dfrac{x}{4}\right)^2 + \left(\dfrac{y}{5}\right)^2 = 1$

So the cartesian equation is $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$.

C8 Use the method of example 7 to show that the general ellipse, defined by $x = a\cos\theta,\ y = b\sin\theta$, has the cartesian equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

C9 An ellipse has the equation $9x^2 + 4y^2 = 36$. Rewrite this in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and hence write down a pair of parametric equations for this ellipse.

C10 Find a pair of parametric equations for each of these ellipses.

(a) $4x^2 + y^2 = 36$ **(b)** $x^2 + 9y^2 = 9$ **(c)** $9x^2 + 0.25y^2 = 2.25$

The method of example 7 can be applied to a circle.

Consider a circle, radius 3 units and centre $(2, -1)$.
It can be defined parametrically as $x = 3\cos\theta + 2,\ y = 3\sin\theta - 1$.

Hence $\cos\theta = \dfrac{x-2}{3},\ \sin\theta = \dfrac{y+1}{3}$

Substituting into $\cos^2\theta + \sin^2\theta = 1$, $\left(\dfrac{x-2}{3}\right)^2 + \left(\dfrac{y+1}{3}\right)^2 = 1$

Hence the cartesian equation of the circle is $(x-2)^2 + (y+1)^2 = 3^2$.
This corresponds to $(x-a)^2 + (y-b)^2 = r^2$, (from Core 1, a circle with centre (a, b) and radius r) with $a = 2$, $b = -1$ and $r = 3$, as given above.

Exercise C (answers p 151)

1 A circle defined is defined by $x = 2\cos\theta,\ y = 2\sin\theta$, where θ is in radians. Give the exact values of the cartesian coordinates of the point where $\theta = \dfrac{\pi}{3}$.

2 Write a pair of parametric equations for each ellipse produced as follows.

(a) From a circle, centre the origin and of unit radius, that has been stretched by a factor of 4 in the x-direction

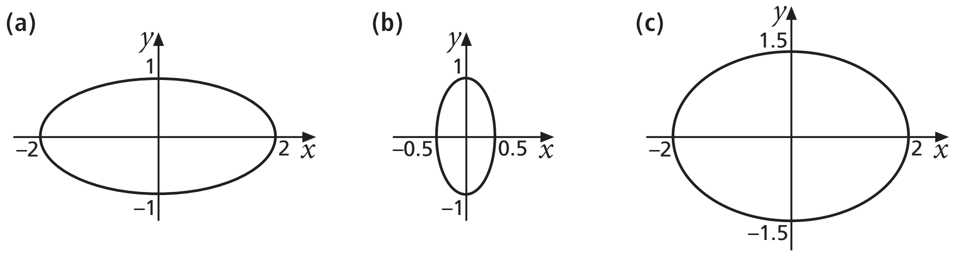
(b) From a circle, centre the origin and of unit radius, that has been 'stretched' by a factor of 0.7 in the y-direction

3 An ellipse is defined by the parametric equations $x = 3\cos\theta,\ y = 4\sin\theta$. Find the exact coordinates of the points where θ has values $0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \pi$ radians.

4 Find the cartesian equation of the ellipse defined by $x = \frac{1}{2}\cos\theta,\ y = \frac{1}{3}\sin\theta$.

5 The curves shown are ellipses. In each case, write

(i) a pair of parametric equations **(ii)** the cartesian equation



6 A circle is defined by the cartesian equation $(x-3)^2+(y+2)^2=25$.

(a) State its centre. **(b)** State its radius. **(c)** Define it parametrically.

***7** A curve is given by $x=2\cos\theta+3,\ y=\sin\theta-1$.

(a) Draw a sketch of it.

(b) Find the cartesian equation for it.

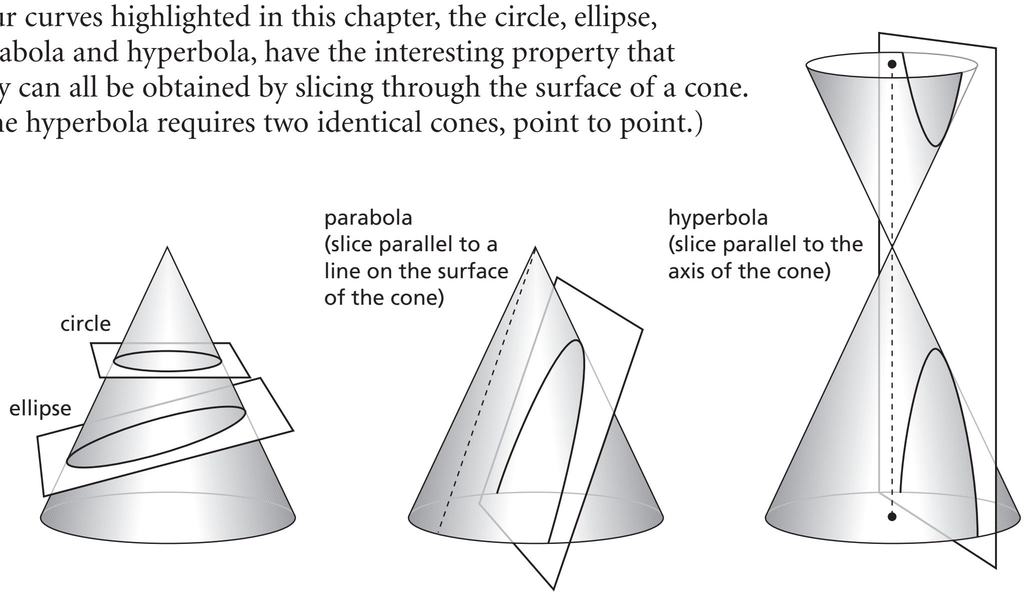
***8** Define the circle $x^2+y^2-6x+8y-11=0$ parametrically.

***9** Obtain a cartesian equation for each of these pairs of parametric equations.

(a) $x=\cot\theta,\ y=2\operatorname{cosec}\theta$ **(b)** $x=2\sec\theta,\ y=3\tan\theta$

Conic sections

Four curves highlighted in this chapter, the circle, ellipse, parabola and hyperbola, have the interesting property that they can all be obtained by slicing through the surface of a cone. (The hyperbola requires two identical cones, point to point.)



The path traced out by a body moving in space under the gravitational influence of a large body (such as our Sun) can be any of these conic sections.

Most planets in the solar system have an orbit that is nearly circular, but Mercury and Pluto have significantly elliptical orbits, while comets follow very stretched-out ellipses. (In elliptical motion of this kind, the large body is not at the centre of the ellipse and the parameter θ does not represent time.)

A body that does not orbit the Sun, but passes by and is deflected by the Sun's gravity, follows a hyperbolic path.

These curves are also used by architects and structural engineers, sometimes for aesthetic reasons but also often because their mathematical properties help with structural stability.

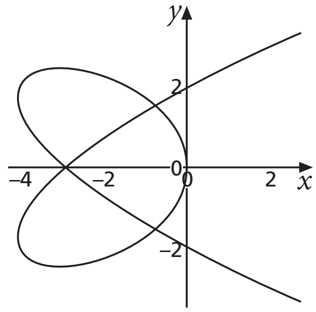
Pattern parametrics

You can produce many interesting curves using parametric equations.
Most of those considered here go beyond what you will meet in the exam.
But it's interesting to feed them into a graph plotter and think about the result.

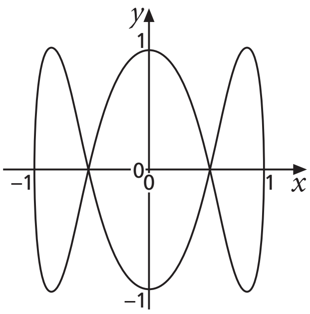
The one shown here uses only simple polynomials:

$$x = t^4 - 4t^2, \quad y = t^3 - 3t \quad (t = -3 \text{ to } 3)$$

(It gets its swooping character because x in terms of t has three stationary points within the stated limits of t, while y in terms of t has two stationary points. Bear this in mind if you want to want to produce something similar of your own.)



Other curves besides a circle or ellipse – such as this one – can be produced using sine and cosine. Try to work out what equations have been used, then check with a graph plotter.



Plot some of these on a graph plotter, which must be set to degrees.
Some of them have established names, which are given. You can make up your own names for the others.

Conjecture what effect a particular modification (such as changing a constant) will have on the curve, then test and see if you were right.

1 $x = \cos 5t, \ y = \sin 3t \ (t = 0 \text{ to } 400)$

2 $x = \sin t - \cos 2t, \ y = \cos t - \sin 2t \ (t = 0 \text{ to } 360)$ (trefoil)

3 $x = \dfrac{t^3}{50}, \ y = \sin 100t \ (t = -10 \text{ to } 10)$

4 $x = 3\cos 90t, \ y = 3\sin 100t \ (t = 0 \text{ to } 40)$

5 $x = \cos t + \cos^2 t, \ y = \sin t + \cos t \sin t \ (t = 0 \text{ to } 360)$ (cardioid)

6 $x = 3(1 - 0.01t)\cos 99t, \ y = 3(1-0.01t)\sin 100t \ (t = 0 \text{ to } 95)$

7 $x = 3\sin t + \sin 12t, \ y = 3\cos t + \cos 12t \ (t = 0 \text{ to } 360)$

8 $x = 3\sin t, \ y = \dfrac{3\cos^2 t(2 + \cos t)}{3 + \sin^2 t} \ (t = 0 \text{ to } 360)$

9 $x = at + \cos 100t$ (where $a = 0.2, 0.5, 1, 2$), $y = \sin 100t \ (t = -10 \text{ to } 10)$

10 $x = 3\sin t + \sin 10t, \ y = 3\cos t + \cos 13t \ (t = 0 \text{ to } 360)$

Key points

- Two functions that separately define the x- and y-coordinates of a graph in terms of a third variable are called parametric equations.
  The third variable is called the parameter. (p 32)
- For a graph defined parametrically, transformations are applied as follows:
  To apply a translation $\begin{bmatrix} a \\ b \end{bmatrix}$ add a to the x-function and b to the y-function.
  (This may be used to give a circle or ellipse a centre other than the origin.)
  To stretch in the x-direction, multiply the x-function by the required factor; correspondingly for a stretch in the y-direction.
  To reflect in the y-axis multiply the x-function by –1; to reflect in the x-axis multiply the y-function by –1. (p 33)
- To convert a pair of parametric equations to a single cartesian equation, eliminate the parameter using methods like those used for simultaneous equations. (pp 35–37)
- The curve $x = r\cos\theta,\ y = r\sin\theta$ is a circle with radius r, centre the origin. (p 39)
- The curve $x = a\cos\theta,\ y = b\sin\theta$ is an ellipse, centre the origin.
  Its width is $2a$ units and its height is $2b$ units.
  Its cartesian equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. (pp 40–41)

Test yourself (answers p 152)

1 A curve is defined by $x = \dfrac{1}{t^2},\ y = 3t\ \ (t \neq 0)$.
Find the coordinates of the points on the curve where $t = -3, -2, -1, 1, 2$ and 3.

2 For each pair of parametric equations below, find the cartesian coordinates for $t = -2, -1, 0, 1, 2$, then plot these points and draw the graph.

(a) $x = t + 1,\ y = 4 - t^2$ **(b)** $x = t^3 - 1,\ y = t^2 + 1$ **(c)** $x = 2 - t,\ y = t^3 + t^2 + t$

3 Find the coordinates of the point(s) where each of the following curves meets the x- and y-axes.

(a) $x = t - 3,\ y = 3t + 1$ **(b)** $x = t^2 - 2t,\ y = 5 - t$

(c) $x = 2t - 8,\ y = t^2 + t - 6$ **(d)** $x = \dfrac{t+1}{t-1},\ y = t^3 + 27\ \ (t \neq 1)$

4 Give the coordinates of the points where the curve defined by $x = t^2 - 5t,\ y = \dfrac{t+1}{t}$ meets the line $x = -6$.

5 Convert each of these pairs of parametric equations to an equation of the form $ax + by + c = 0$.

(a) $x = 4t + 5,\ y = 3t$ **(b)** $x = 5 - 2t,\ y = \frac{1}{3}t$

6 Obtain a single cartesian equation for each pair of parametric equations.

(a) $x = 3t,\ y = t^2$

(b) $x = t^3,\ y = \frac{1}{2}t$

(c) $x = \sqrt{t},\ y = t^3\ \ (t \geq 0)$

(d) $x = t - 1,\ y = 2 - t^2$

(e) $x = \dfrac{1}{t},\ y = 3t\ \ (t \neq 0)$

(f) $x = t^2 - t,\ y = \frac{1}{4}t$

(g) $x = t + 2,\ y = t^3 + 2t$

(h) $x = \dfrac{1}{2t},\ y = 3 - t\ \ (t \neq 0)$

(i) $x = 3t^3 + 2,\ y = 5 - t^3$

(j) $x = t(3 - t),\ y = \sqrt{t}\ \ (t \geq 0)$

(k) $x = t^2,\ y = \dfrac{1}{t - 2}\ \ (t \neq 2)$

(l) $x = \dfrac{t}{t - 1},\ y = \dfrac{t}{t + 1}\ \ (t \neq -1, t \neq 1)$

(m) $x = 3\sqrt{t} - 2,\ y = 4 - \sqrt{t}\ \ (t \geq 0)$

(n) $x = \dfrac{t}{2t - 3},\ y = \dfrac{t}{t + 1}\ \ (t \neq \frac{3}{2}, t \neq -1)$

7 A curve is defined parametrically by $x = t^2 + \dfrac{2}{t},\ y = t^2 - \dfrac{2}{t}\ \ (t \neq 0)$.
By first expressing $x + y$ and $x - y$ in terms of t, find the cartesian equation.

8 Describe the curve given by each of these pairs of parametric equations.

(a) $x = 4\cos\theta,\ y = 4\sin\theta$

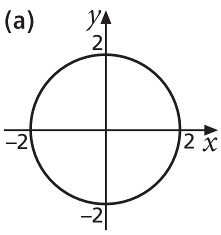
(b) $x = \cos\theta,\ y = 3\sin\theta$

(c) $x = 0.6\cos\theta,\ y = 0.6\sin\theta$

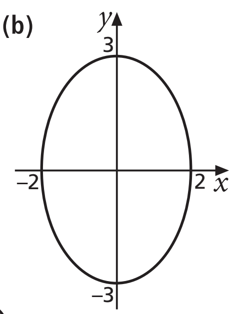
(d) $x = 2\cos\theta,\ y = 3\sin\theta$

9 Give a pair of parametric equations for each of these.

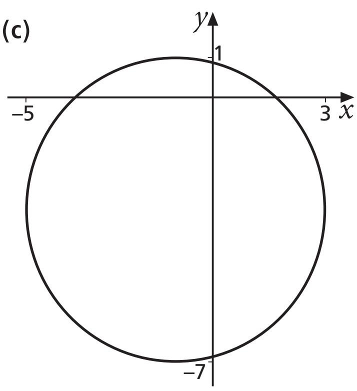
(a)



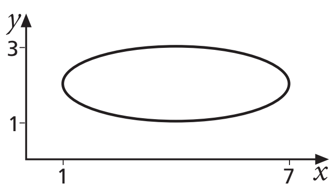
(b)



(c)



(d)



10 An ellipse is defined by the parametric equations $x = 3\cos\theta,\ y = 4\sin\theta$.
Find its cartesian equation.

11 Find a pair of parametric equations for each of these ellipses.

(a) $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$

(b) $25x^2 + 4y^2 = 100$

12 An ellipse is defined by the parametric equations $x = 6\cos\theta,\ y = 2\sin\theta$.
A chord of the ellipse has its ends at points on the ellipse where $\theta = \frac{1}{6}\pi$ and $\frac{5}{6}\pi$ respectively. Find the length of the chord, giving your answer as an exact value.