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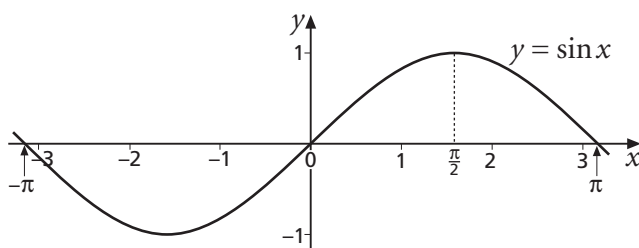
4 Trigonometry

In this chapter you will learn about

- the inverse functions of cosine, sine and tangent
- the reciprocal functions secant, cosecant and cotangent
- transformations of the above functions
- identities and equations involving the above functions

A Inverse circular functions (answers p 171)

This is part of the graph of $y = \sin x$, where x is measured in radians.



A1 Using the same scale on each axis, make a sketch copy of the graph above.

Draw the line $y = x$ on your sketch.

Draw the reflection of $y = \sin x$ in the line $y = x$ on your sketch.

Mark carefully the position of the reflection of the point $(\frac{\pi}{2}, 1)$.

A2 You saw in chapter 1 that reflecting the graph of a function in the line $y = x$ may give the graph of an inverse function.

Does it give an inverse function in this case? If not, why not?

In order that the reflection of $y = \sin x$ in the line $y = x$ may represent a function, we need to restrict the domain of $y = \sin x$, so that $y = \sin x$ is a one-one function.

A3 Which of these are one-one functions?

$$f(x) = \sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \quad g(x) = \sin x, \quad 0 \leq x \leq \pi \quad h(x) = \sin x, \quad \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$$

How we restrict the domain of $y = \sin x$ so it has an inverse is to some extent arbitrary, but the convention is that we restrict x to the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

The inverse function of $\sin x$ is denoted by $y = \sin^{-1} x$.

(Note that $\sin^{-1} x$ never means $\frac{1}{\sin x}$.)

The range of $\sin^{-1} x$ is the restricted domain of $\sin x$, that is $-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$.

We call the range of $\sin^{-1} x$ the **principal values** of $\sin^{-1} x$.

So, for example, $\sin^{-1} 1 = \frac{\pi}{2}$ ($\frac{\pi}{2}$ is the principal value for the angle whose sine is 1).

A4 What is the domain of $\sin^{-1} x$?

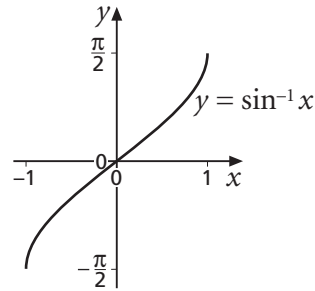
The graph of $y = \sin^{-1} x$ is shown on the right.

We also write $\arcsin x$ or $\text{asin } x$ for $\sin^{-1} x$.

K

The domain of $y = \sin^{-1} x$ is $-1 \leq x \leq 1$.

The range of $y = \sin^{-1} x$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.



A5 Using degrees, $\sin^{-1} 1 = 90^\circ$. In degrees, write down the value of

(a) $\sin^{-1} \frac{1}{2}$

(b) $\sin^{-1} \left(\frac{1}{\sqrt{2}} \right)$

(c) $\sin^{-1} 0$

(d) $\sin^{-1} (-1)$

A6 In order for $\tan x$ to have an inverse, we restrict its domain to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

(a) With this domain, what is the range of $\tan x$?

(b) Using equal scales on the x - and y -axes, sketch the graph of $y = \tan x$ with domain as stated above.

(c) On the same graph, sketch and label the inverse function of $\tan x$, which is $y = \tan^{-1} x$.

(d) Write down the domain and range of $y = \tan^{-1} x$.

Example 1

Write down, in degrees, the value of

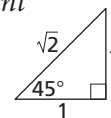
(a) $\sin^{-1} \left(-\frac{1}{\sqrt{2}} \right)$

(b) $\tan^{-1} \left(-\frac{1}{\sqrt{3}} \right)$

Solution

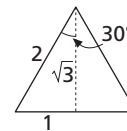
You need to know, or to be able quickly to work out, the sine, cosine and tangent of 0° , 30° , 45° , 60° and 90° .

(a) $\sin 45^\circ = \frac{1}{\sqrt{2}}$, so $\sin(-45^\circ) = -\frac{1}{\sqrt{2}}$.



$\sin^{-1} x$ is between -90° and 90° . Hence $\sin^{-1} \left(-\frac{1}{\sqrt{2}} \right) = -45^\circ$.

(b) $\tan 30^\circ = \frac{1}{\sqrt{3}}$, so $\tan(-30^\circ) = -\frac{1}{\sqrt{3}}$.



$\tan^{-1} x$ is between -90° and 90° . Hence $\tan^{-1} \left(-\frac{1}{\sqrt{3}} \right) = -30^\circ$.

Exercise A (answers p 171)

- 1** In order for $\cos x$ to have an inverse, we restrict its domain to $0 \leq x \leq \pi$.
- (a) Using equal scales on the x - and y -axes, sketch the graph of $y = \cos x$ with the above domain.
- (b) On the same graph, sketch the inverse function of $\cos x$, $y = \cos^{-1} x$.
- (c) Write down the domain and range of $y = \cos^{-1} x$.
- 2** (a) Write down the solution in degrees, between 0° and 90° , of $\cos \theta^\circ = \frac{1}{2}$.
- (b) Hence write down the two solutions to $\cos \theta^\circ = -\frac{1}{2}$ between 0° and 360° .
- (c) What is the principal value of $\cos^{-1}(-\frac{1}{2})$ in degrees?
- 3** Write down the value, in degrees, of each of the following
- (a) $\cos^{-1}(-1)$ (b) $\cos^{-1}(\frac{\sqrt{3}}{2})$ (c) $\cos^{-1} 0$ (d) $\tan^{-1}(-\sqrt{3})$
- 4** Write down, in radians, the value of each of these.
- (a) $\cos^{-1} 1$ (b) $\cos^{-1}(-\frac{1}{2})$ (c) $\tan^{-1}(-1)$ (d) $\sin^{-1}(-\frac{\sqrt{3}}{2})$
- (e) $\cos^{-1} 0$ (f) $\tan^{-1} 0$ (g) $\cos^{-1}(-\frac{1}{\sqrt{2}})$ (h) $\sin^{-1}(-\frac{1}{2})$
- 5** Solve these where possible.
- (a) $\sin^{-1} x = \frac{\pi}{4}$ (b) $\sin^{-1} x = \frac{\pi}{6}$ (c) $\sin^{-1} x = -\frac{\pi}{4}$ (d) $\sin^{-1} x = \pi$
- 6** (a) Work these out.
- (i) $\sin(\sin^{-1} \frac{1}{2})$ (ii) $\tan(\tan^{-1} 1)$ (iii) $\cos(\cos^{-1} \frac{\sqrt{3}}{2})$ (iv) $\sin(\sin^{-1} -\frac{1}{2})$
- (b) Is it always true that $\sin(\sin^{-1} x) = x$?
- 7** (a) Work these out.
- (i) $\sin^{-1}(\sin \frac{\pi}{6})$ (ii) $\sin^{-1}(\sin \frac{5\pi}{6})$ (iii) $\sin^{-1}(\sin \frac{7\pi}{6})$ (iv) $\sin^{-1}(\sin \frac{13\pi}{6})$
- (b) Is it always true that $\sin^{-1}(\sin x) = x$?
- 8** (a) Which of the following domains ensures that the function $f(x) = \sin 2x$ is one-one?
- A $-\pi \leq x \leq \pi$ B $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ C $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$
- (b) With the chosen domain, what is the range of $f(x)$?
- (c) Write down the domain and range of the inverse function, $f^{-1}(x)$.
- *9** (a) Using your answers to question 8, suggest a possible inverse function for $y = \sin 2x$.
- (b) Check your suggestion, using a graph plotter, by
- (i) reflecting the graph of $y = \sin 2x$ in the line $y = x$
- (ii) plotting the graph of your suggestion and checking that it coincides with (i).

B Sec, cosec and cot (answers p 171)

The most useful trigonometrical functions are cosine, sine and tangent. However there are three other closely related functions – secant, cosecant and cotangent (usually abbreviated to sec, cosec and cot).

These are defined as

$$\sec x = \frac{1}{\cos x} \qquad \operatorname{cosec} x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x}$$

B1 (a) When $x = \frac{\pi}{4}$, what is the value of $\cos x$?

(b) What is the value of $\sec x$ when $x = \frac{\pi}{4}$?

B2 What is the value of $\sec x$ when

(a) $x = \frac{\pi}{6}$ (b) $x = \frac{\pi}{3}$ (c) $x = 0$ (d) $x = -\frac{\pi}{4}$

B3 Write down the value of $\operatorname{cosec} x$ when

(a) $x = \frac{\pi}{2}$ (b) $x = -\frac{\pi}{2}$ (c) $x = \frac{\pi}{4}$ (d) $x = -\frac{\pi}{6}$

B4 Evaluate these.

(a) $\cot \frac{\pi}{4}$ (b) $\cot \frac{\pi}{6}$ (c) $\cot\left(-\frac{\pi}{3}\right)$ (d) $\cot \frac{5\pi}{4}$

D

B5 (a) What happens when you try to evaluate $\operatorname{cosec} 0$?

(b) Write down two other values of x for which $\operatorname{cosec} x$ is not defined.

(c) Describe accurately the complete set of values of x for which $\operatorname{cosec} x$ is not defined.

B6 (a) Write down three values of x for which $\sec x$ is not defined.

(b) Describe clearly the complete set of values of x for which $\sec x$ is not defined.

The diagram shows part of the graph of $\cos x$.

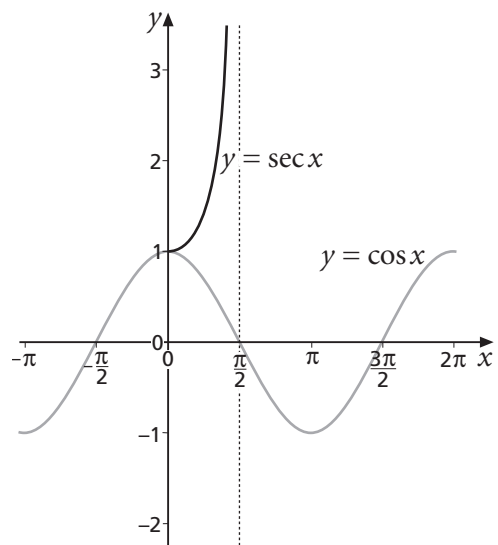
Part of the graph of $\sec x$ is also shown.

B7 (a) Copy the diagram and complete the graph of $y = \sec x$ for $-\pi \leq x \leq 2\pi$.

(b) Describe clearly the domain of $\sec x$ (you will need to use your answer to B6).

(c) Write down an expression for the range of $y = \sec x$.

(d) What is the period of $\sec x$?



- B8** (a) Sketch the graph of $\sin x$.
 (b) On your diagram, sketch the graph of $\operatorname{cosec} x$.
 (c) Write down expressions for the domain and range of $\operatorname{cosec} x$.
 (d) What is the period of $\operatorname{cosec} x$?

The cotangent of x , $\cot x$, is defined as $\frac{1}{\tan x}$.

Using the relationship between $\sin x$ and $\cos x$, we can establish a relationship between $\sec x$ and $\tan x$.

$$\sin^2 x + \cos^2 x = 1$$

Dividing by $\cos^2 x$ we obtain $\frac{\sin^2 x}{\cos^2 x} + 1 = \frac{1}{\cos^2 x}$

$$\Rightarrow \tan^2 x + 1 = \sec^2 x$$

- B9** Divide the relationship $\sin^2 x + \cos^2 x = 1$ by $\sin^2 x$ to obtain a relationship between $\operatorname{cosec} x$ and $\cot x$.

Example 2

Show that $\frac{\tan \theta}{1 + \tan^2 \theta} = \sin \theta \cos \theta$.

Solution

$$\frac{\tan \theta}{1 + \tan^2 \theta} = \frac{\tan \theta}{\sec^2 \theta} = \frac{\tan \theta}{\frac{1}{\cos^2 \theta}} = \tan \theta \times \cos^2 \theta = \frac{\sin \theta}{\cos \theta} \times \cos^2 \theta = \sin \theta \cos \theta$$

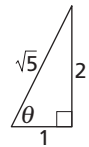
Example 3

Given that $\tan \theta = 2$, find exact values for (a) $\cot \theta$ (b) $\sec \theta$ (c) $\operatorname{cosec} \theta$

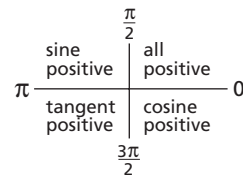
Solution

(a) $\cot \theta = \frac{1}{\tan \theta} = \frac{1}{2}$

One possible pair of values for $\sin \theta$ and $\cos \theta$ is $\sin \theta = \frac{2}{\sqrt{5}}$ and $\cos \theta = \frac{1}{\sqrt{5}}$.



However, both sine and cosine could be negative (for an angle between π and $\frac{3\pi}{2}$).



In this case, $\sin \theta = -\frac{2}{\sqrt{5}}$ and $\cos \theta = -\frac{1}{\sqrt{5}}$. Hence

(b) $\sec \theta = \frac{1}{\cos \theta} = \frac{1}{\frac{1}{\sqrt{5}}}$ or $\frac{1}{-\frac{1}{\sqrt{5}}} = \sqrt{5}$ or $-\sqrt{5}$

(c) $\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{1}{\frac{2}{\sqrt{5}}}$ or $\frac{1}{-\frac{2}{\sqrt{5}}} = \frac{\sqrt{5}}{2}$ or $-\frac{\sqrt{5}}{2}$

Example 4Simplify $\operatorname{cosec}\left(\frac{\pi}{2} - x\right)$.**Solution**

$$\operatorname{cosec}\left(\frac{\pi}{2} - x\right) = \frac{1}{\sin\left(\frac{\pi}{2} - x\right)} = \frac{1}{\cos x} = \sec x$$

K

$$\sec x = \frac{1}{\cos x}$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

$$\operatorname{cosec}^2 x = 1 + \cot^2 x$$

$$\cot x = \frac{1}{\tan x}$$

Exercise B (answers p 172)

- Sketch the graph of $\tan x$ for $-2\pi < x < 2\pi$.
 - On your diagram, sketch also the graph of $\cot x$.
 - Give the domain and range of $\cot x$, clearly stating any values that are excluded.
 - What is the period of $\cot x$?
- Find the exact values of
 - $\operatorname{cosec} \frac{\pi}{3}$
 - $\cot \frac{\pi}{3}$
 - $\sec\left(-\frac{\pi}{3}\right)$
 - $\operatorname{cosec}\left(-\frac{\pi}{3}\right)$
- Simplify these.
 - $\operatorname{cosec} x \tan x$
 - $\tan x \cot x$
 - $\sin x \operatorname{cosec} x$
 - $\sec x \sin x$
- Simplify these.
 - $\frac{\cos \theta}{\sin \theta}$
 - $\frac{\operatorname{cosec} \theta}{\sec \theta}$
 - $\sec\left(\frac{\pi}{2} - x\right)$
 - $\cot(\pi + x)$
- Simplify the following.
 - $\frac{\sin x}{1 + \cot^2 x}$
 - $\cos x \sqrt{1 + \tan^2 x}$
- Prove the following identities.
 - $\tan \theta + \cot \theta = \sec \theta \operatorname{cosec} \theta$
 - $\cot \theta \sec \theta = \operatorname{cosec} \theta$
- Simplify these.
 - $\operatorname{cosec}^2 x - 1$
 - $(\sec x + 1)(\sec x - 1)$
 - $\frac{\tan x}{1 - \sec^2 x}$
- Show that $\operatorname{cosec} x + \cot x = \frac{1}{\operatorname{cosec} x - \cot x}$ ($\operatorname{cosec} x \neq \cot x$).
- Given that $\sin \theta = \frac{4}{5}$, and that θ is in the first quadrant, find exact values of
 - $\cos \theta$
 - $\tan \theta$
 - $\sec \theta$
 - $\operatorname{cosec} \theta$
- Given that $\frac{\pi}{2} \leq \theta \leq \pi$ and $\cot \theta = -2\frac{2}{5}$, find exact values of
 - $\tan \theta$
 - $\cos \theta$
 - $\sin \theta$
 - $\operatorname{cosec} \theta$
- Given that $\tan \theta = 1$, what are the possible values of
 - $\cot \theta$
 - $\sin \theta$
 - $\cos \theta$
 - $\operatorname{cosec} \theta$

C Solving equations (answers p 173)

When solving equations involving $\sec x$, $\operatorname{cosec} x$ or $\cot x$, it is often best to substitute so that the equation involves $\sin x$, $\cos x$ or $\tan x$ instead.

You may also be able to make use of the relations $\sec^2 x = 1 + \tan^2 x$ and $\operatorname{cosec}^2 x = 1 + \cot^2 x$.

Example 5

Solve the equation $\sec x = 2$ for $0 \leq x \leq 2\pi$.

Solution $\sec x = 2$
Substitute for $\sec x$. $\frac{1}{\cos x} = 2$
 $\Rightarrow \cos x = \frac{1}{2}$
 $\Rightarrow x = \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

Example 6

Solve $\cot^2 \theta^\circ = 3$, where $-180^\circ \leq \theta^\circ \leq 180^\circ$.

Solution $\cot^2 \theta^\circ = 3$
Substitute $\frac{1}{\tan \theta^\circ}$ for $\cot \theta^\circ$. $\frac{1}{\tan^2 \theta^\circ} = 3$
 $\Rightarrow \tan^2 \theta^\circ = \frac{1}{3}$
 $\Rightarrow \tan \theta^\circ = \pm \frac{1}{\sqrt{3}}$

If $\tan \theta^\circ = \frac{1}{\sqrt{3}}$, then $\theta^\circ = 30^\circ$ or -150° ; if $\tan \theta^\circ = -\frac{1}{\sqrt{3}}$, then $\theta^\circ = -30^\circ$ or 150° .

Hence $\theta^\circ = -150^\circ, -30^\circ, 30^\circ$ or 150° .

Example 7

Solve the equation $\tan^2 x + \sec x = 1$ for $0 \leq x \leq 2\pi$.

Solution $\tan^2 x + \sec x = 1$
You need to get a quadratic in a single variable, so substitute $\tan^2 x = \sec^2 x - 1$.
 $\sec^2 x - 1 + \sec x = 1$
 $\Rightarrow \sec^2 x + \sec x - 2 = 0$
Factorise. $(\sec x + 2)(\sec x - 1) = 0$
 $\Rightarrow \sec x = -2$ or $\sec x = 1$
 $\Rightarrow \cos x = -\frac{1}{2}$ or $\cos x = 1$
 $\Rightarrow x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or $x = 0$ or 2π

- C1** Solve $\tan^2 x + \sec x = 1$ by substituting $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$ as the first step, and then solving a quadratic in $\cos x$.

Check your answer is identical to that in example 7.

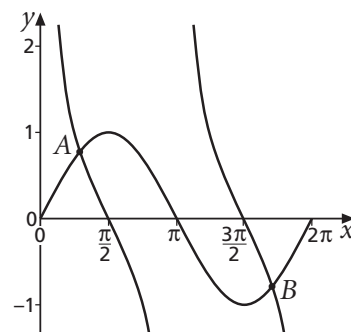
Exercise C (answers p 173)

- 1** (a) Solve $\sec \theta^\circ = \sqrt{2}$, giving exact answers within the range 0° to 360° .
 (b) Hence solve $\sec^2 \theta^\circ = 2$ ($0^\circ \leq \theta^\circ \leq 360^\circ$).
- 2** Solve $\cot \theta^\circ = 3$, giving answers in the range -180° to 180° to the nearest degree.
- 3** Solve for $0 \leq x \leq 2\pi$
 (a) $\cot x = 1$ (b) $\operatorname{cosec} x = 1$ (c) $\operatorname{cosec}^2 x = 1$ (d) $3 \operatorname{cosec}^2 x = 4$
- 4** Solve, giving answers in the range $-\pi$ to π to two decimal places,
 (a) $\sec x = 3$ (b) $\sec^2 x = 9$ (c) $\operatorname{cosec} x = 3$ (d) $\cot x = \frac{1}{2}$
- 5** (a) By replacing $\sec^2 x$ by $1 + \tan^2 x$, show that the equation

$$\sec^2 x + \tan x = 3$$
 is equivalent to the equation

$$\tan^2 x + \tan x - 2 = 0$$

 (b) Factorise the left-hand side of this equation.
 (c) Solve the equation to find all the values of x between $-\pi$ and π .
- 6** By substituting for $\cot^2 \theta$ in terms of $\operatorname{cosec}^2 \theta$, solve $2 \cot^2 \theta + \operatorname{cosec} \theta + 1 = 0$ for $-\pi \leq \theta \leq \pi$.
- 7** Solve $2 \tan^2 x - 7 \sec x + 5 = 0$, giving answers between 0 and 2π to one decimal place.
- 8** Solve $3 \cot x + 2 \tan x = 5$. Give answers between 0° and 90° to one decimal place.
- 9** Solve the following equations for $0^\circ \leq \theta^\circ \leq 360^\circ$.
 Give your answers to two decimal places.
 (a) $2 \sec^2 \theta^\circ = 9 \tan \theta^\circ + 7$ (b) $\operatorname{cosec}^2 \theta^\circ = 3 \cot \theta^\circ + 5$
- 10** Solve $\sec x = \tan x$ for $0 \leq x \leq 2\pi$.
 Explain your result with the aid of a graph.
- 11** The diagram shows the graphs of $y = \sin x$ and $y = \cot x$ ($0 \leq x \leq 2\pi$).
 Find the coordinates of the points A and B, to two decimal places.



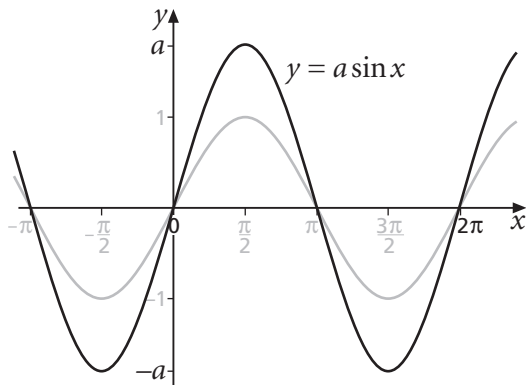
D Transforming graphs (answers p 173)

In chapter 3 you saw how transforming the graph of a function affects its equation.

The diagrams below show how $y = \sin x$ (shown in grey) is changed after the transformations shown.

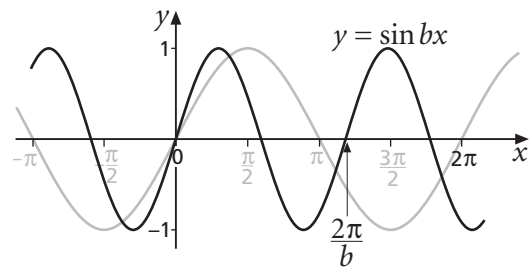
$y = a \sin x$:

stretch in the y -direction, scale factor a

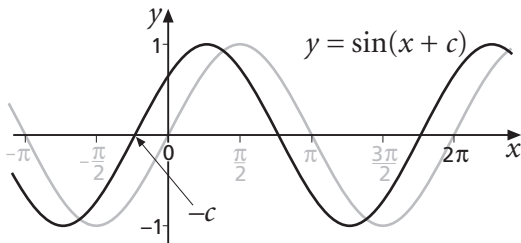


$y = \sin bx$:

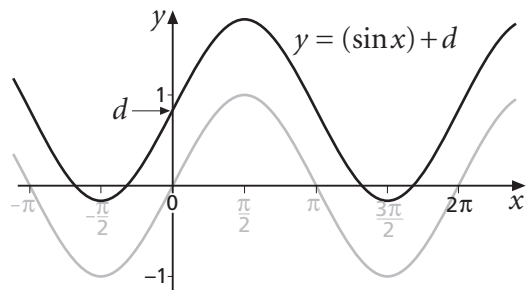
stretch in the x -direction, scale factor $\frac{1}{b}$



$y = \sin(x + c)$: a translation by $\begin{bmatrix} -c \\ 0 \end{bmatrix}$

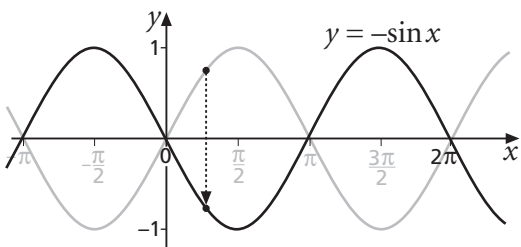


$y = (\sin x) + d$: a translation by $\begin{bmatrix} 0 \\ d \end{bmatrix}$



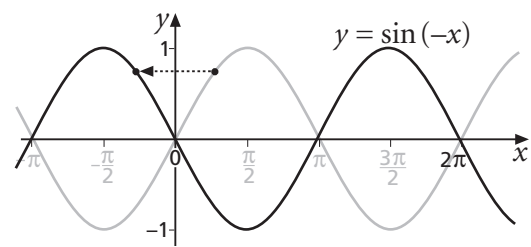
$y = -\sin x$:

reflection in the x -axis



$y = \sin(-x)$:

reflection in the y -axis



D1 Without doing any plotting, state the two transformations that will transform $y = \sin x$ on to each of these.

(a) $y = \frac{1}{3} \sin(-x)$

(b) $y = \sin(2x) + 3$

(c) $y = 3 - \sin x$ (note: $3 - \sin x = -(\sin x) + 3$)

D2 (a) What is the equation of the image of $y = \cos x$ after each of the following?

(i) A translation by $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, followed by a stretch in the x -direction, scale factor 4

(ii) A stretch in the y -direction, scale factor 2, followed by a translation by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

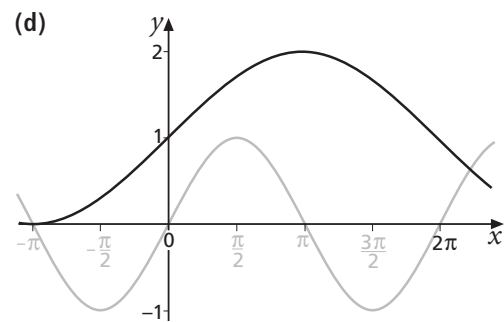
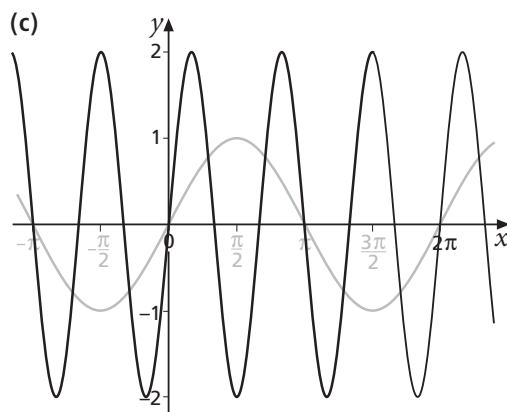
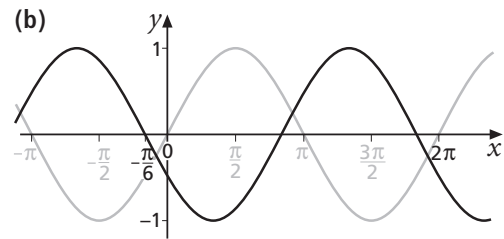
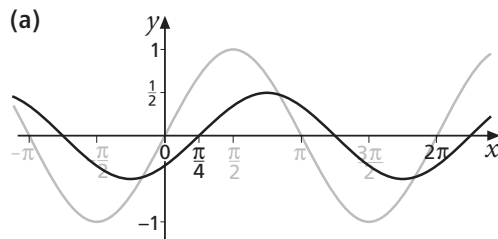
(iii) A stretch in the x -direction, scale factor 3, followed by a stretch in the y -direction, scale factor 2

(b) For each part of (a), sketch $y = \cos x$ and the result of the transformations.

D3 For each of the graphs below,

(i) state two transformations that will transform $y = \sin x$ (shown in grey on each diagram) on to the given graph

(ii) hence write down the equation of the graph



Example 8

Sketch the graph of $y = \cos \theta^\circ$ ($-360^\circ \leq \theta \leq 360^\circ$), and its image after a stretch in the θ -direction, scale factor 2, followed by a stretch in the y -direction, scale factor 1.5.

What is the equation of the image?

Solution

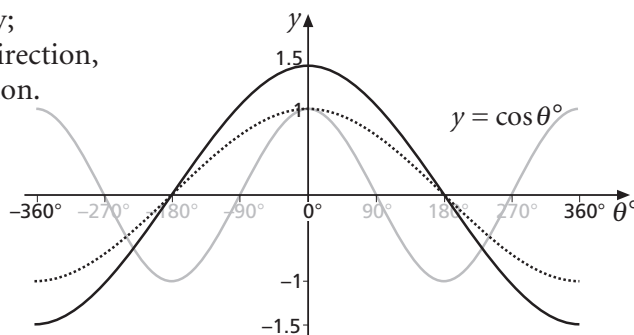
The graph of $y = \cos \theta^\circ$ is shown in grey; shown dotted after the stretch in the θ -direction, and solid after the stretch in the y -direction.

The graph of $y = \cos \theta^\circ$ becomes $y = \cos \frac{1}{2}\theta^\circ$ after a stretch in the θ -direction, scale factor 2.

The graph of $y = \cos \frac{1}{2}\theta^\circ$ becomes $y = 1.5 \cos \frac{1}{2}\theta^\circ$ after a stretch in the y -direction, scale factor 1.5.

The final graph has the equation $y = 1.5 \cos \frac{1}{2}\theta^\circ$.

It is sensible to now check this by drawing the graph of $y = 1.5 \cos \frac{1}{2}\theta^\circ$ on a graph plotter and seeing whether it agrees with the graph you have sketched.



Example 9

State two transformations which, when applied to the graph of $y = \tan x$, will give the graph of $y = \tan 2x - 1$. Hence sketch the graph of $y = \tan 2x - 1$, where $-2\pi \leq x \leq 2\pi$.

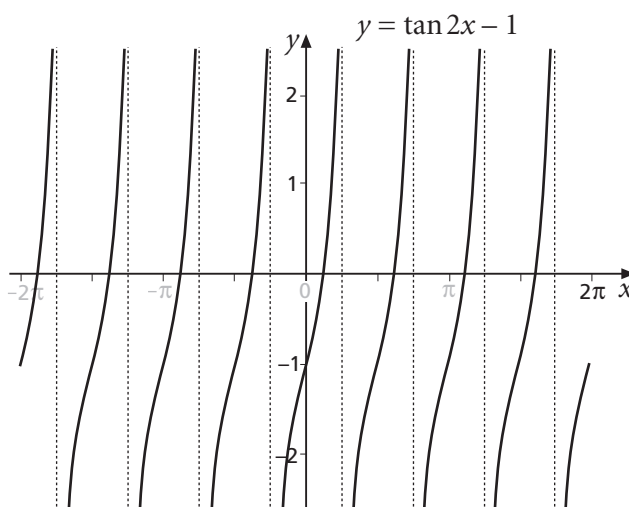
Solution

The graph of $y = \tan 2x - 1$ can be obtained from that of $y = \tan x$ by first applying a stretch in the x -direction, scale factor $\frac{1}{2}$ (giving $y = \tan 2x$)

and then by applying the translation $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ to $y = \tan 2x$ (giving $y = \tan 2x - 1$).

Hence the graph is

It is sensible to check this on a graph plotter.



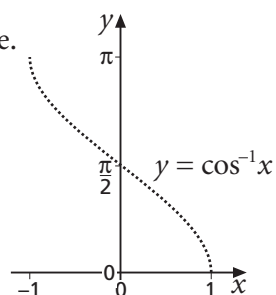
Example 10

Sketch the graph of $y = \cos^{-1} 2x$. State its domain and range.

Solution

Start by sketching the graph of $y = \cos^{-1} x$.

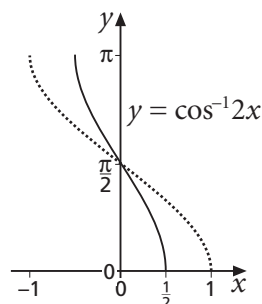
The graph of $y = \cos^{-1} x$ is shown dotted on the right.



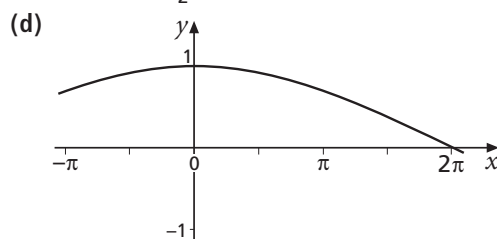
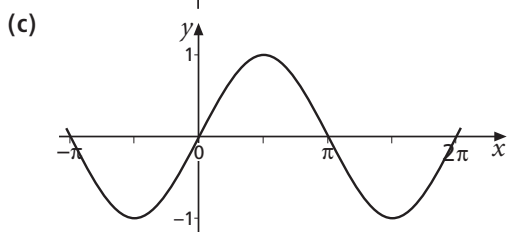
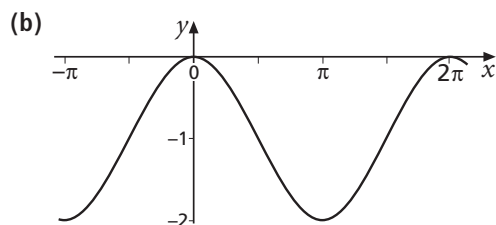
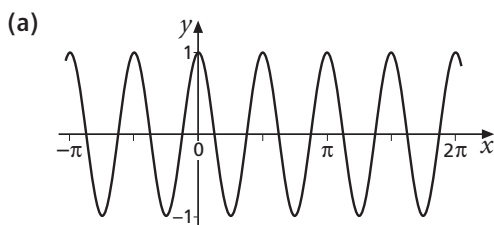
We obtain the graph of $y = \cos^{-1} 2x$ from that of $y = \cos^{-1} x$ by replacing x by $2x$, that is by applying a stretch in the x -direction with scale factor $\frac{1}{2}$.

Transforming $y = \cos^{-1} x$ in this way we obtain the graph of $y = \cos^{-1} 2x$ (shown solid).

The domain is $-\frac{1}{2} \leq x \leq \frac{1}{2}$, and the range is $0 \leq y \leq \pi$.

**Exercise D** (answers p 174)

- 1 Each of these graphs shows $y = \cos x$ after a single transformation. Identify the transformation and thus the equation of each graph.

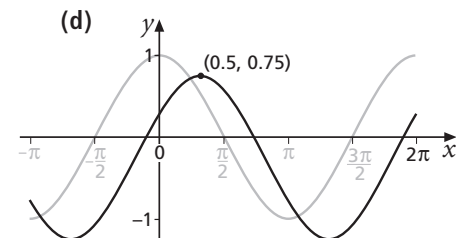
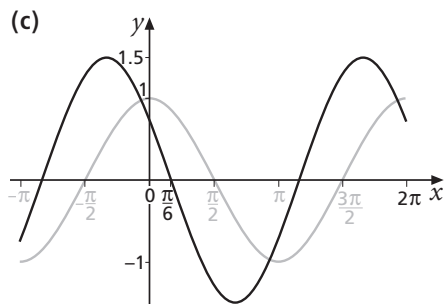
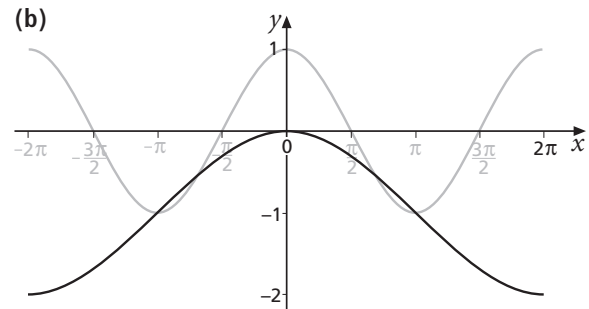
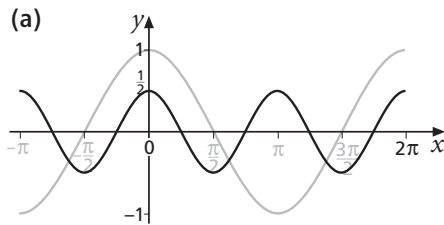


- 2 (a) Find the equation of the resulting graph when $y = \tan x$ is transformed by
- a stretch, scale factor 2, in the x -direction, followed by a translation by $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$
 - a reflection in the y -axis, followed by a stretch, factor $\frac{1}{2}$, in the x -direction
 - a translation by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ followed by reflection in the x -axis
 - a translation by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ followed by reflection in the y -axis
- (b) For each part of (a), sketch $y = \tan x$ and the result of the transformations.

- 3 (a) The point $A\left(\frac{\pi}{6}, \frac{1}{2}\right)$ lies on the graph of $y = \sin x$.
 What are the coordinates of the image of A after a stretch, factor 2,
 in the y -direction, followed by a translation by $\begin{bmatrix} \frac{\pi}{4} \\ 0 \end{bmatrix}$?
- (b) The graph of $y = \sin x$ is transformed by the pair of transformations in (a).
 What is the equation of the transformed graph?
- (c) Check that the coordinates of the image of A satisfy the equation
 of the transformed graph.

4 For each of the graphs below,

- (i) state two transformations that will transform $y = \cos x$
 (shown in grey on each diagram) on to the given graph
- (ii) hence write down the equation of the graph



5 For $-\pi \leq x \leq 2\pi$ sketch the graphs of

- (a) $y = \sin x$ (b) $y = |\sin x|$ (c) $y = \sin|x|$

Check using a graph plotter.

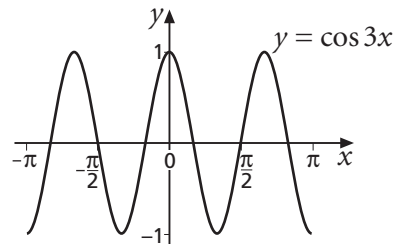
6 (a) Sketch the graph of $y = \sec x$ for $-\pi \leq x \leq \pi$.

(b) On the same diagram, sketch the graph of $y = \sec x$ after a stretch in the y -direction,
 scale factor $\frac{1}{2}$, followed by a stretch in the x -direction, scale factor $\frac{1}{3}$.

(c) What is the equation of the resulting graph?
 Check with a graph plotter.

7 The graph of $y = \cos 3x$ is shown ($-\pi \leq x \leq \pi$).

- (a) Identify the coordinates of each maximum and minimum of $y = \cos 3x$ in the interval $-\pi \leq x \leq \pi$.
- (b) Copy the graph and on your copy draw the graph of $y = |\cos 3x|$.
- (c) Use your answers to (a) to solve the equation $|\cos 3x| = 1$ ($-\pi \leq x \leq \pi$).



- 8 (a) Find the equation of the resulting graph when $y = \sin^{-1} x$ is first stretched, scale factor 2, in the y -direction and then translated by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- (b) Sketch the resulting graph.
Check using a graph plotter.

9 $f(x)$ is defined by $f(x) = \sin \frac{1}{2}x$ for $x \in \mathbb{R}$, $-\pi \leq x \leq \pi$;

$g(x)$ is defined by $g(x) = |x|$ for $x \in \mathbb{R}$.

- (a) State the range of $f(x)$.
- (b) State the domain and range of $f^{-1}(x)$.
- (c) Which of the following functions is $f^{-1}(x)$?

$f^{-1}(x) = \sin^{-1} 2x$

$f^{-1}(x) = \sin^{-1} \frac{1}{2}x$

$f^{-1}(x) = 2 \sin^{-1} x$

$f^{-1}(x) = \frac{1}{2} \sin^{-1} x$

- (d) Write down an expression for $fg(x)$.
- (e) Sketch $y = fg(x)$.

*10 Use a graph plotter for this question.

- (a) Draw the graph of $y = \tan x$ and $y = \cot x$ on the same axes.
- (b) The two graphs appear to be the same shape.
Perform two transformations on the graph of $y = \cot x$ to superimpose the graph on to that of $y = \tan x$ to confirm this observation.
- (c) Write down the equation of the graph of $y = \cot x$ after the two transformations in part (b).
- (d) Prove algebraically that your equation in (c) is identical to $y = \tan x$.

E Order of transformations (answers p 176)

In section D, the order in which two transformations were applied to (for example) $y = \sin x$ did not in general matter.

This will always be the case if one transformation affects only x , and the other transformation affects only y , as you saw in chapter 3.

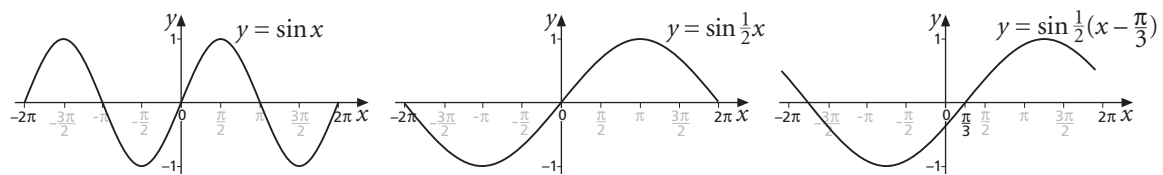
However, consider transforming $y = \sin x$ by applying first a stretch of factor 2 in the x -direction, then a translation by $\begin{bmatrix} \frac{\pi}{3} \\ 0 \end{bmatrix}$.

After a stretch of factor 2 in the x -direction, x is replaced by $\frac{1}{2}x$, so $y = \sin x$ becomes $y = \sin \frac{1}{2}x$.

After a translation by $\begin{bmatrix} \frac{\pi}{3} \\ 0 \end{bmatrix}$, x is replaced by $x - \frac{\pi}{3}$,

so $y = \sin \frac{1}{2}x$ becomes $y = \sin \frac{1}{2}(x - \frac{\pi}{3})$, which is $y = \sin(\frac{1}{2}x - \frac{\pi}{6})$.

The sequence of graphs is



- E1** (a) Sketch a sequence of graphs for when the transformations above are applied to $y = \sin x$ in the opposite order.
 (b) What is the equation of the image of $y = \sin x$ after the transformations?

Example 11

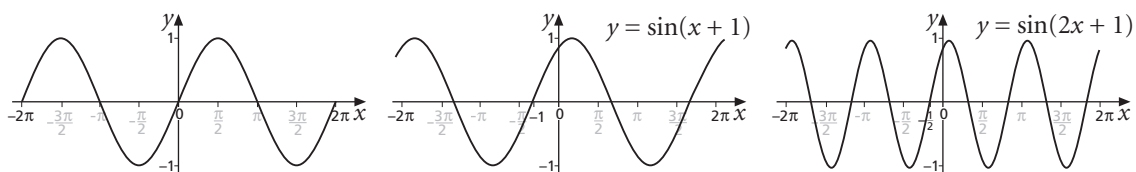
Find a sequence of transformations that will transform the graph of $y = \sin x$ on to the graph of $y = \sin(2x + 1)$. Thus sketch $y = \sin(2x + 1)$.

Solution

First $y = \sin x$ is transformed on to $y = \sin(x + 1)$ by replacing x by $x + 1$, that is by a translation by $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Then $y = \sin(x + 1)$ is transformed on to $y = \sin(2x + 1)$ by replacing x by $2x$, that is by a stretch, scale factor $\frac{1}{2}$, in the x -direction.

The sequence of graphs is



- E2** The transformed curve in example 11 passes through $(-\frac{1}{2}, 0)$.
 How could you transform $y = \sin x$ on to $y = \sin(2x + 1)$,
 where one of the transformations is a translation by $\begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$?

Example 12

The graph of $y = \sin x$ is first translated by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and then stretched by a factor of $\frac{1}{3}$ in the y -direction.

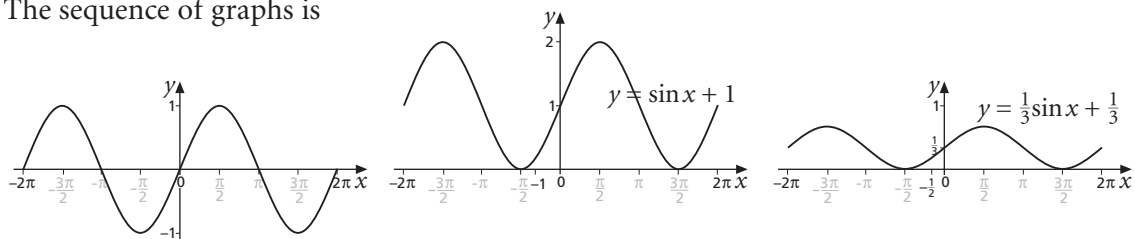
What is the equation of the resulting graph?

Solution

A translation by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has the effect of replacing y by $(y - 1)$,
 so $y = \sin x$ becomes $y - 1 = \sin x$, which is $y = \sin x + 1$.

A stretch in the y -direction, scale factor $\frac{1}{3}$, has the effect of replacing y by $3y$.
 So $y = \sin x + 1$ becomes $3y = \sin x + 1$, which is $y = \frac{1}{3}\sin x + \frac{1}{3}$.

The sequence of graphs is



- E3** (a) What is the resulting equation if the transformations in example 12 are applied in the opposite order?
 (b) Sketch the sequence of graphs arising from applying the transformations in this order.

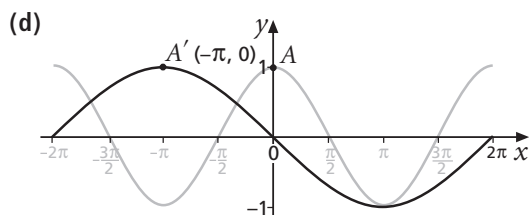
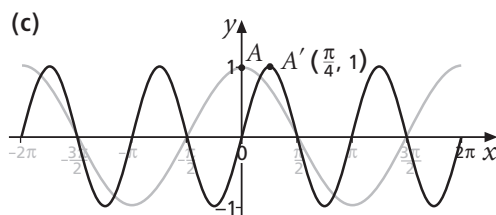
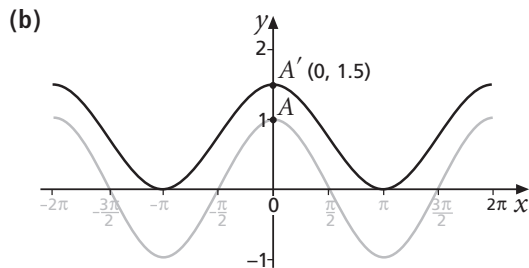
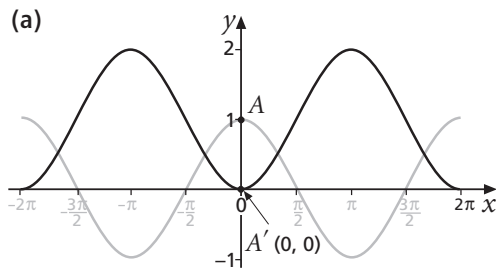
Exercise E (answers p 177)

- 1** (a) What is the equation of the resulting graph if the graph of $y = \cos x$ is first stretched in the x -direction, factor $\frac{1}{2}$, and then translated by $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$?
 (b) What is the equation of the graph if the two transformations are applied in the reverse order?
- 2** Find the equation of the image of the graph of $y = \tan x$ after
- a translation by $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ followed by reflection in the y -axis
 - a reflection in the y -axis followed by a stretch in the x -direction, factor 2
 - a stretch in the x -direction, factor $\frac{1}{2}$, followed by a translation by $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$
 - a translation by $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ followed by a stretch in the x -direction, factor 3
 - a reflection in the y -axis followed by a translation by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

- 3** Find the equation of the image of the graph of $y = \cos x$ after
- a stretch in the y -direction, factor 2, followed by a translation of $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$
 - a translation of $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ followed by reflection in the x -axis
 - a reflection in the x -axis followed by a translation of $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
 - a translation of $\begin{bmatrix} 0 \\ -3 \end{bmatrix}$ followed by a stretch in the y -direction, factor $\frac{1}{3}$
 - a reflection in the x -axis followed by a stretch in the y -direction, factor 5

- 4** (a) State a pair of transformations, making the order clear, which will map
- $y = \sin \theta$ on to $y = \sin(3\theta - 1)$
 - $y = \sin \theta$ on to $y = \sin 2(\theta - 3)$
 - $y = \cos \theta$ on to $y = 2 \cos \theta + 1$
 - $y = \cos \theta$ on to $y = \frac{1}{2}(\cos \theta - 1)$
- (b) For each part of (a), sketch the original graph and the result of the transformations. Check with a graph plotter.

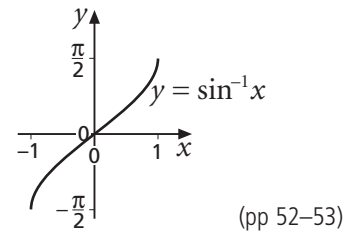
- 5** On each diagram below, the graph of $y = \cos x$ is shown in grey. The image of $y = \cos x$ after two transformations is shown in black. The image of the point $A(0, 1)$ is A' , and is shown on each diagram. Identify a pair of transformations for each diagram, and hence the equation of the transformed curve.



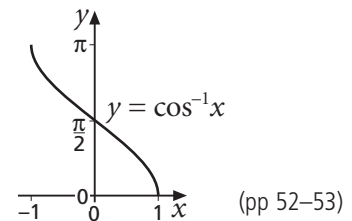
- 6** Two functions are defined as $f(x) = \operatorname{cosec} x$ and $g(x) = |x|$.
- On separate graphs, sketch $fg(x)$ and $gf(x)$ for $-3\pi \leq x \leq 3\pi$.
 - Does $fg(x) = gf(x)$?
 - Suggest a domain that will make $fg(x) = gf(x)$.

Key points

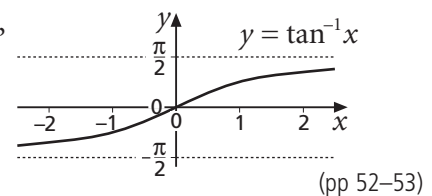
- The inverse function of $\sin x$ is denoted by $y = \sin^{-1} x$, $\arcsin x$ or $\text{asin } x$, and has domain $-1 \leq x \leq 1$ and range $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.



- The inverse function of $\cos x$ is denoted by $y = \cos^{-1} x$, $\arccos x$ or $\text{acos } x$, and has domain $-1 \leq x \leq 1$ and range $0 \leq y \leq \pi$.



- The inverse function of $\tan x$ is denoted by $y = \tan^{-1} x$, $\arctan x$ or $\text{atan } x$, and has domain $x \in \mathbb{R}$ and range $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

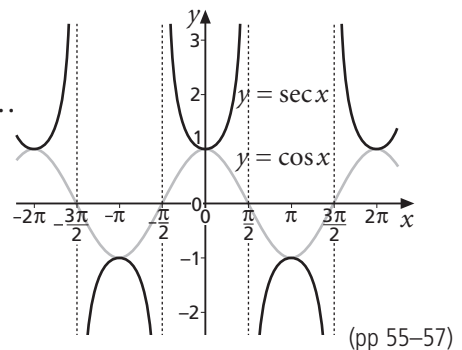


- $\sec x$ is defined as $\frac{1}{\cos x}$.

$y = \sec x$ has domain $x \in \mathbb{R}$, $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$

and range $y \leq -1$ and $y \geq 1$;

its period is 2π .

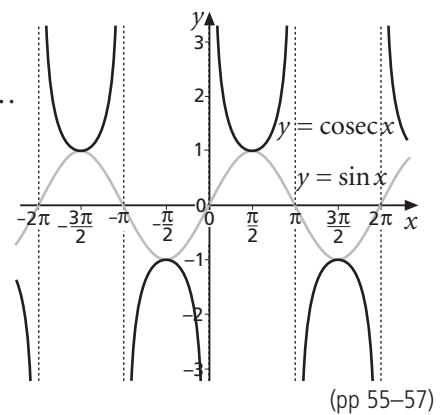


- $\text{cosec } x$ is defined as $\frac{1}{\sin x}$.

$y = \text{cosec } x$ has domain $x \in \mathbb{R}$, $x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$

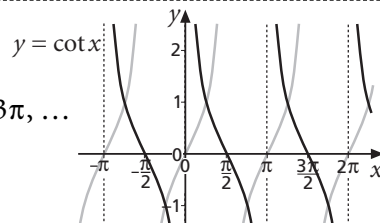
and range $y \leq -1$ and $y \geq 1$;

its period is 2π .



- $\cot x$ is defined as $\frac{1}{\tan x}$.

$y = \cot x$ has domain $x \in \mathbb{R}, x \neq 0, \pm\pi, \pm2\pi, \pm3\pi, \dots$
and range $y \in \mathbb{R}$; its period is π .
($\tan x$ is shown in grey on the graph.)



(pp 55–57)

- $\sec^2 x = 1 + \tan^2 x$

(p 57)

- $\operatorname{cosec}^2 x = 1 + \cot^2 x$

(p 57)

Test yourself (answers p 178)

1 Write down, in radians, the exact value of

(a) $\sin^{-1} \frac{1}{2}$ (b) $\tan^{-1} \left(\frac{1}{\sqrt{3}} \right)$ (c) $\cos^{-1} \left(-\frac{\sqrt{3}}{2} \right)$ (d) $\cos^{-1} \left(-\frac{1}{2} \right)$

2 Give exact values of

(a) $\operatorname{cosec} 30^\circ$ (b) $\sec 45^\circ$ (c) $\cot 60^\circ$ (d) $\sec (-60^\circ)$

3 Simplify these.

(a) $\cot x \sec x$ (b) $\frac{1}{(1 - \operatorname{cosec} x)(1 + \operatorname{cosec} x)}$ (c) $\frac{\sec^2 x}{(\sec x - 1)(1 + \sec x)}$

4 Solve $\operatorname{cosec}^2 x + \cot x - 1 = 0$ giving answers as multiples of π in the range 0 to 2π .

5 Solve the equation $3 \cot^2 x + 8 \operatorname{cosec} x + 1 = 0$,
giving all values of x to the nearest degree in the interval $0^\circ \leq x \leq 360^\circ$.

6 State the two transformations that will map $y = \tan x$ on to

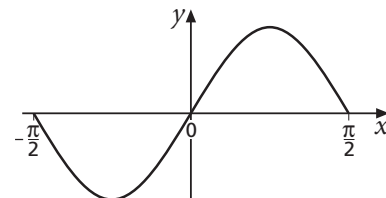
(a) $y = 2 \tan(-x)$ (b) $y = \tan(2x) + 1$ (c) $y = -\tan\left(\frac{1}{2}x\right)$ (d) $y = 2 - \tan x$

7 What is the equation of the image of $y = \sin x$ after each of these?

- (a) A stretch in the x -direction, factor 4, followed by a translation by $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$
(b) A reflection in the y -axis, followed by a stretch in the y -direction, factor 2
(c) A translation by $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ followed by a reflection in the x -axis

8 The diagram shows a sketch of the curve with
equation $y = \sin 2x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

(a) Find, in radians, the values of x in the interval
 $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ for which $\sin 2x = -\frac{1}{5}$.



(b) (i) Draw on a single diagram sketches of the
graphs with equations $y = |x|$ and $y = |\sin 2x|$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

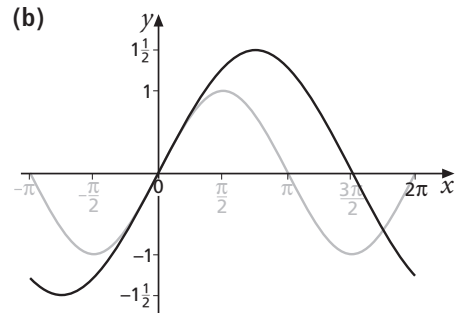
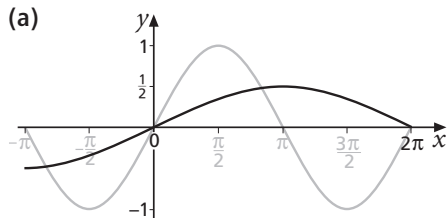
(ii) Hence state the number of times the graph of the curve with
equation $y = |\sin 2x| - |x|$ intersects the x -axis in the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

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9 For each of the graphs below,

(i) state the two transformations that will transform $y = \sin x$ on to the given graph ($y = \sin x$ is shown in grey on each diagram)

(ii) hence write down the equation of the graph



10 Find the equation of the resulting curve if $y = \sin x$ is

(a) reflected in the x -axis and then translated by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(b) translated by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and then reflected in the x -axis

(c) stretched in the x -direction, factor $\frac{1}{3}$, and then translated by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(d) translated by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and then stretched in the x -direction, factor $\frac{1}{3}$

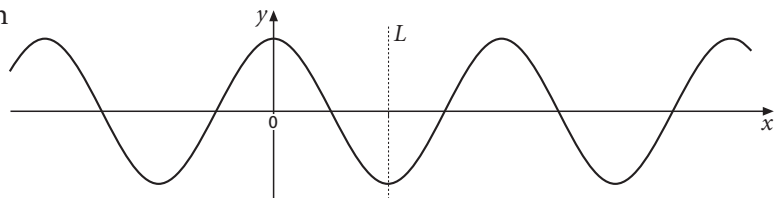
11 Find a pair of transformations, making the order in which they are performed clear, which will map $y = \operatorname{cosec} x$ on to

(a) $y = \operatorname{cosec}(\frac{1}{2}x + 1)$

(b) $y = \operatorname{cosec} 3(x - 2)$

(c) $y = 2(\operatorname{cosec} x) - 3$

12 The diagram shows a sketch of the graph of $y = \cos 2x$ with a line of symmetry L .



(a) (i) Describe the geometrical transformation by which the graph of $y = \cos 2x$ can be obtained from that of $y = \cos x$.

(ii) Write down the equation of the line L .

The function f is defined for the restricted domain $0 \leq x \leq \frac{\pi}{2}$ by $f(x) = \cos 2x$.

(b) (i) State the range of the function f .

(ii) Write down the domain and range of the inverse function f^{-1} , making it clear which is the domain of f^{-1} and which is its range.

(iii) Sketch the graph of $y = f^{-1}(x)$.

The function g is defined for all real numbers by $g(x) = |x|$.

(c) (i) Write down an expression for $gf(x)$.

(ii) Sketch the graph of $y = gf(x)$.