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5 Binomial expansion

In this chapter you will learn how to find any term in the expansion of $(1 + x)^n$.

A Pascal's triangle (answers p 142)

To multiply out or **expand** $(a + b)^2$, you can use a table:

$$\begin{array}{r|cc} & a & b \\ a & a^2 & ab \\ b & ab & b^2 \end{array} \quad (a + b)^2 = a^2 + 2ab + b^2$$

Similarly you can expand $(a + b)^3$ by multiplying $(a^2 + 2ab + b^2)$ by $(a + b)$:

$$\begin{array}{r|ccc} & a^2 & 2ab & b^2 \\ a & a^3 & 2a^2b & ab^2 \\ b & a^2b & 2ab^2 & b^3 \end{array} \quad (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

- A1** (a) Expand $(a + b)^4$ by multiplying $(a^3 + 3a^2b + 3ab^2 + b^3)$ by $(a + b)$.
 (b) Verify that $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$.

Notice these things about the expansion of $(a + b)^5$:

- The terms start with a^5 ; in each following term the power of a goes down by 1 and the power of b goes up by 1, until b^5 is reached. The total power is always 5.
- The coefficients are 1, 5, 10, 10, 5, 1; they form a symmetrical pattern.

If you detach the coefficients from the expansions, you get an interesting pattern:

$$\begin{array}{l} (a + b)^0 = \mathbf{1} \\ (a + b)^1 = \mathbf{1}a + \mathbf{1}b \\ (a + b)^2 = \mathbf{1}a^2 + \mathbf{2}ab + \mathbf{1}b^2 \\ (a + b)^3 = \mathbf{1}a^3 + \mathbf{3}a^2b + \mathbf{3}ab^2 + \mathbf{1}b^3 \\ (a + b)^4 = \mathbf{1}a^4 + \mathbf{4}a^3b + \mathbf{6}a^2b^2 + \mathbf{4}ab^3 + \mathbf{1}b^4 \\ (a + b)^5 = \mathbf{1}a^5 + \mathbf{5}a^4b + \mathbf{10}a^3b^2 + \mathbf{10}a^2b^3 + \mathbf{5}ab^4 + \mathbf{1}b^5 \end{array} \quad \begin{array}{cccccc} & & & & & \mathbf{1} \\ & & & & & \mathbf{1} & \mathbf{1} \\ & & & & \mathbf{1} & \mathbf{2} & \mathbf{1} \\ & & & \mathbf{1} & \mathbf{3} & \mathbf{3} & \mathbf{1} \\ & & \mathbf{1} & \mathbf{4} & \mathbf{6} & \mathbf{4} & \mathbf{1} \\ \mathbf{1} & \mathbf{5} & \mathbf{10} & \mathbf{10} & \mathbf{5} & \mathbf{1} \end{array}$$

The triangular pattern of numbers is called **Pascal's triangle**, after the French mathematician Blaise Pascal (1623–63).

- A2** (a) How are the numbers in each row of Pascal's triangle related to the numbers in the previous row?
 (b) Write down the next row of the triangle and hence expand $(a + b)^6$.
 (c) Can you explain the relationship between the rows of Pascal's triangle? (You might do so using the 'table' method of expansion above.)

Example 1Expand $(1 + 2x)^4$.**Solution**

This comes from the expansion of $(a + b)^4$, with a replaced by 1, and b by $2x$.

The expansion starts with 1^4 .

The next term includes 1^3 and $(2x)^1$.

After this comes a term with 1^2 and $(2x)^2$, and so on.

The coefficients, from Pascal's triangle, are 1, 4, 6, 4, 1.

$$\begin{aligned} \text{The expansion is } (1 + 2x)^4 &= 1^4 + 4 \times 1^3 \times (2x)^1 + 6 \times 1^2 \times (2x)^2 + 4 \times 1^1 \times (2x)^3 + (2x)^4 \\ &= 1 + 4 \times 2x + 6 \times 4x^2 + 4 \times 8x^3 + 16x^4 \\ &= 1 + 8x + 24x^2 + 32x^3 + 16x^4 \end{aligned}$$

Example 2Expand $(1 - 3x)^3$.**Solution**

This comes from the expansion of $(a + b)^3$, with a replaced by 1, and b by $-3x$.

$$\begin{aligned} \text{The expansion is } (1 - 3x)^3 &= 1^3 + 3 \times 1^2 \times (-3x)^1 + 3 \times 1^1 \times (-3x)^2 + (-3x)^3 \\ &= 1 - 9x + 27x^2 - 27x^3 \end{aligned}$$

Exercise A (answers p 142)

Use Pascal's triangle (extended if necessary) to answer these questions.

- 1 Expand $(1 + x)^3$.
- 2 Expand each of these.
(a) $(1 + 2x)^3$ (b) $(1 + 2x)^5$ (c) $(1 - x)^6$ (d) $(1 - 3x)^6$ (e) $(1 + \frac{1}{2}x)^7$
- 3 Expand (a) $(1 - \frac{1}{2}x)^5$ (b) $(2 + x)^3$ (c) $(3 - 2x)^4$
- 4 (a) Expand $(1 + x)^8$ up to and including the term in x^3 .
(b) By substituting $x = 0.1$ in your result for (a), calculate an approximate value for $(1.1)^8$.
(c) By means of a suitable substitution, use your result for (a) to calculate an approximate value for $(0.9)^8$.

If n is small, then the expansion of $(a + b)^n$ can be found by extending Pascal's triangle as far as is necessary.

However, for large values of n this method is cumbersome. What is needed is a way of calculating the coefficients for a given value of n . This is provided by the **binomial theorem**. In order to understand it, you need to know something about arrangements.

B Arrangements (answers p 142)

Suppose you have four letters: A, B, C, D. They can be arranged in order in different ways, for example ACDB, CBDA, DBAC, and so on. (Another word for an arrangement in order is 'permutation'.)

The first letter can be A, B, C or D. Each of these 4 choices can be followed by any of the 3 others, so these are the possible choices for the first two letters:

A then B, C or D: AB... AC... AD...

B then A, C or D: BA... BC... BD...

C then A, B or D: CA... CB... CD...

D then A, B or C: DA... DB... DC...

There are thus 12 choices for the first two letters. This is because there are 4 ways of choosing the first letter and each choice can be combined with 3 ways of choosing the second letter.

Each of these 12 possibilities can be followed by either of the remaining 2 letters. For example, AB can be followed by either C or D, giving ABC..., ABD... So altogether there are 24 choices for the first three letters.

Finally, for the fourth letter there is only one choice – the last remaining letter. For example, ABD must be followed by C, giving ABDC. So there are also 24 arrangements of all four letters.

- B1** (a) How many arrangements are there of the five letters A, B, C, D, E?
(b) How can you work out the number of arrangements without having to list them all and count them?

Suppose you have seven letters to arrange in order. There are 7 choices for the first letter. Each of these can be combined with 6 choices for the second letter, and so on:

$$\text{Number of arrangements of 7 letters} = \begin{matrix} 1\text{st} & 2\text{nd} & 3\text{rd} & 4\text{th} & 5\text{th} & 6\text{th} & 7\text{th} \\ 7 & \times & 6 & \times & 5 & \times & 4 & \times & 3 & \times & 2 & \times & 1 \end{matrix} = 5040$$

The number $7 \times 6 \times 5 \times \dots \times 2 \times 1$ is called **7 factorial** (written $7!$).

In general,

K $n! = n(n-1)(n-2)(n-3) \times \dots \times 3 \times 2 \times 1$

- B2** (a) Calculate the value of $6!$.
(b) Given that $9! = 362\,880$, calculate $10!$.

So far you have been looking at arrangements of a set of different letters. Now suppose that the letters are not all different, for example A, A, B, C.

Here are some of the arrangements: AACB, BAAC, CABA, ...

- B3** (a) How many arrangements are there of the letters A, A, B, C?
(b) Why is the number half the number of arrangements of four different letters?

To see the effect of making some letters identical, first distinguish the identical letters. Think of the set A, A, B, C as A_1, A_2, B, C .

There are $4! = 24$ ways of arranging the four letters A_1, A_2, B, C . Here are some of them:

$A_1A_2BC \quad A_2A_1BC \quad A_1BA_2C \quad A_2BA_1C \quad BCA_1A_2 \quad BCA_2A_1 \quad \dots$

Notice that these arrangements come in pairs, for example A_1A_2BC and A_2A_1BC . As the two letter As are really the same, both arrangements in this pair correspond to the same arrangement AABC.

So the number of arrangements when the two As are identical is $\frac{4!}{2} = 12$.

- D B4** Suppose you have the set A, A, A, B, C. First distinguish the As as A_1, A_2, A_3 .
- (a) How many arrangements are there of the five different letters A_1, A_2, A_3, B, C ?
 - (b) Here is one of the arrangements: $A_1BA_3A_2C$.
How many arrangements (including this one) correspond to the arrangement ABAAC?
 - (c) How many arrangements are there of the set A, A, A, B, C?

Suppose three letters in a set of six are identical: A, A, A, B, C, D.

If the three As are distinguished, then the set becomes A_1, A_2, A_3, B, C, D and there are $6!$ arrangements.

These arrangements can be split up into groups according to the positions occupied by the three As. For example, one group is shown on the right. This group corresponds to the arrangement AABDAC.

$A_1A_2BDA_3C$
$A_1A_3BDA_2C$
$A_2A_1BDA_3C$
$A_2A_3BDA_1C$
$A_3A_1BDA_2C$
$A_3A_2BDA_1C$

There are $3!$ arrangements in each group because there are $3!$ ways of arranging the 3 As within the group.

So when the As are identical, the number of arrangements is $\frac{6!}{3!} = 120$.

The general formula is this:

The number of arrangements of a set of n objects when r of them are identical is $\frac{n!}{r!}$.

Arrangements with objects of only two kinds

The special case that will be needed for the binomial theorem is where the set consists of n objects (letters) of which r are of one kind and the remaining $n - r$ are of another.

For example, suppose the set is A, A, A, A, B, B, B.

If all the As and all the Bs are distinguished, then there are $7!$ arrangements altogether.

Because the four As are identical, you have to divide by $4!$. This gives $\frac{7!}{4!}$ arrangements in which the As are identical.

But because the three Bs are identical, you then have to divide this number by $3!$.

Number of arrangements of A, A, A, A, B, B, B = $\frac{7!}{4! \times 3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1)}$

Notice that many of the factors in the numerator and denominator cancel out.

After cancelling out as many as possible, the result is $\frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$.

The number of ways of arranging n objects of which r are of one type and $n - r$ of another is denoted by the symbol $\binom{n}{r}$. Its value is given by $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

In practice, many factors in the numerator and denominator cancel out.

For example, $\binom{8}{3} = \frac{8!}{3! \times 5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1)} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1}$.

This leads to another formula:

K

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots}{r!}$$

Continue until you have r factors in the numerator.

So $\binom{n}{1} = n$ $\binom{n}{2} = \frac{n(n-1)}{2!}$ $\binom{n}{3} = \frac{n(n-1)(n-2)}{3!}$

Written in this way, the general formula is

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

The r th factor here is $n - (r - 1)$ or $n - r + 1$.

Your calculator may use a different notation: ${}_n C_r$ or ${}^n C_r$. To calculate $\binom{7}{3}$, enter 7, press ${}_n C_r$ and then enter 3 = .

The letter 'C' in this notation stands for 'combination', which arises as follows. Suppose you have 7 people and you want to select a group of 3 from them, where the order of selection does not matter. Each 'combination' of 3 people corresponds to saying 'yes' (Y) to 3 people and 'no' (N) to the other 4 people. Here, for example, is one possible combination:

Person: A B C D E F G
Decision: Y N N Y N Y N

Each combination corresponds to an arrangement of 3 Ys and 4 Ns.

So the number of combinations is $\binom{7}{3}$.

B5 Use one of the formulae above to calculate $\binom{9}{3}$. Then check using a calculator.

B6 Use a calculator to find the value of (a) $\binom{6}{2}$ (b) $\binom{9}{4}$ (c) $\binom{20}{10}$

B7 (a) $\binom{7}{0}$ is the number of arrangements of 7 objects, of which none are of one type and 7 are of the other. So what is its value?

(b) If you use the factorial formula to work out $\binom{7}{0}$, what value must you give to 0!?

C The binomial theorem (answers p 142)

You should now be in a position to see how the work on arrangements relates to expanding expressions of the form $(a + b)^n$.

Consider $(a + b)^2 = (a + b)(a + b)$.

Normally when you expand this you get $a^2 + 2ab + b^2$.

However, for the time being distinguish between ab and ba .

ab arises from multiplying a in the first bracket by b in the second.

ba arises from multiplying b in the first bracket by a in the second.

$$\begin{array}{c} ab \\ \overbrace{(a + b)(a + b)} \\ \underbrace{ba} \end{array}$$

Write a^2 as aa and b^2 as bb .

So $(a + b)^2$ is expanded as $aa + ab + ba + bb$.

C1 $(a + b)^3 = (aa + ab + ba + bb)(a + b)$.

Write out the expansion of $(a + b)^3$ in the form $aaa + aab + \dots$

(In other words, don't simplify, combine terms or use indices.)

The expansion of $(a + b)^4$ is the result of multiplying out $(a + b)(a + b)(a + b)(a + b)$.

Each term arises from multiplying as from some brackets by bs from the others.

For example, this combination $\underbrace{(a + b)(a + b)(a + b)(a + b)}$ gives the term $abba$.

The expansion of $(a + b)^4$, written without any simplification or index notation, is as follows:

$$\begin{aligned} (a + b)^4 = & aaaa \\ & + aaab + aaba + abaa + baaa \\ & + aabb + abab + abba + baab + baba + bbaa \\ & + abbb + babb + bbab + bbba \\ & + bbbb \end{aligned}$$

The first term is of course a^4 .

The second line contains all the terms that correspond to a^3b .

They consist of all the arrangements of 1 b and 3 as , so the coefficient of a^3b is $\binom{4}{1} = 4$.

The next line contains all the terms that correspond to a^2b^2 .

They consist of all the arrangements of 2 bs and 2 as , so the coefficient of a^2b^2 is $\binom{4}{2} = 6$.

Similarly for the other two lines.

So $(a + b)^4 = a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + b^4$

This expansion of $(a + b)^4$ is an example of the **binomial theorem**:

$$(a + b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \dots + b^n$$

An important special case arises by letting $a = 1$ and $b = x$:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + x^n$$

which can also be written:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + x^n$$

The values of the binomial coefficients can be found either from a calculator or by using one of the formulae. If the value of n is small, you could also find the coefficients by extending Pascal's triangle.

C2 (a) Write down the first four terms in the expansion of $(1+x)^{10}$.

(b) By replacing x by $-x$, write down the first four terms in the expansion of $(1-x)^{10}$.

Example 3

Find the coefficient of y^4 in the expansion of $(1+3y)^9$.

Solution

The term containing y^4 is $\binom{9}{4}(3y)^4 = 126 \times 81y^4 = 10\,206y^4$.

So the coefficient of y^4 is 10 206.

Example 4

Use the binomial theorem to find the value of $(1.02)^8$ correct to two decimal places.

Solution

Use the expansion of $(1+x)^n$ with $x = 0.02$ and $n = 8$. Continue the expansion until the terms become so small that they do not affect the second decimal place.

$$\begin{aligned} (1.02)^8 &= 1 + \binom{8}{1} \times (0.02) + \binom{8}{2} \times (0.02)^2 + \binom{8}{3} \times (0.02)^3 + \dots \\ &= 1 + 8 \times 0.02 + 28 \times 0.0004 + 56 \times 0.000\,008 + \dots \\ &= 1 + 0.16 + 0.0112 + 0.000\,448 + \dots = 1.17 \text{ (to 2 d.p.)} \end{aligned}$$

Example 5

Find the coefficient of x^2 in the expansion of $(1+4x)^4(1-2x)^6$.

Solution

Expand each factor as far as the x^2 term. $(1+4x)^4 = 1 + 16x + 96x^2 + \dots$
 $(1-2x)^6 = 1 - 12x + 60x^2 - \dots$

Pick out the terms that will give an x^2 when multiplied.

$$(1 + \underbrace{16x + 96x^2})(1 - 12x + 60x^2) \quad \text{Term in } x^2 \text{ is } 60x^2 - 192x^2 + 96x^2 = -36x^2$$

So the coefficient of x^2 is -36 .

Example 6

Find the term containing x^3 in the expansion of $(3 - 2x)^7$.

Solution

Use the binomial theorem for $(a + b)^n$ with $a = 3$, $b = -2x$ and $n = 7$.

The term containing x^3 is $\binom{7}{3} \times 3^4 \times (-2x)^3 = 35 \times 81 \times (-8x^3) = -22\,680x^3$.

Exercise C (answers p 142)

- 1 Write down the first four terms in the expansion of $(1 + x)^{12}$.
- 2 Find the coefficient of
(a) x^5 in the expansion of $(1 + x)^{10}$ (b) x^3 in the expansion of $(1 + x)^{20}$
- 3 Use the binomial theorem to find the value of $(1.03)^7$ correct to two decimal places.
- 4 Find the first four terms in the expansion of
(a) $(1 + 2x)^9$ (b) $(1 - 3x)^8$ (c) $(1 + \frac{1}{2}x)^{15}$ (d) $(1 - \frac{1}{5}x)^7$
- 5 (a) Expand $(1 - 2x)^5$ as far as the term in x^2 .
(b) Hence find the coefficient of x^2 in the expansion of $(3 - 4x - x^2)(1 - 2x)^5$.
- 6 Find the coefficient of x^3 in the expansion of
(a) $(2 + x)^9$ (b) $(2 - x)^9$ (c) $(2 + 3x)^9$
- 7 In the expansion of $(1 + kx)^n$, where k and n are positive integers, the coefficient of x^3 is twice the coefficient of x^2 .
(a) Prove that $n = 2 + \frac{6}{k}$. (b) Find all the possible pairs of values of k and n .

Key points

- $n!$ (n factorial) is $n(n-1)(n-2)(n-3) \times \dots \times 3 \times 2 \times 1$ (p 62)
- $(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + x^n$
The coefficient of x^r in this expansion is $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$ (pp 64, 66)

Test yourself (answers p 142)

- 1 Write down the first four terms in the expansion of $(1 + x)^{20}$.
- 2 Find the coefficient of x^3 in the expansion of (a) $(1 + 4x)^7$ (b) $(1 - 2x)^{10}$
- 3 Expand $(1 + 2x)^5(1 - \frac{1}{2}x)^8$ as far as the term in x^2 .