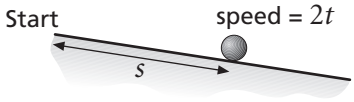
Core 1 for AQA contents

10 Integration

In this chapter you will learn
- what integration is and how it is related to differentiation
- how to integrate polynomial functions

A Thinking backwards (answers p 187)

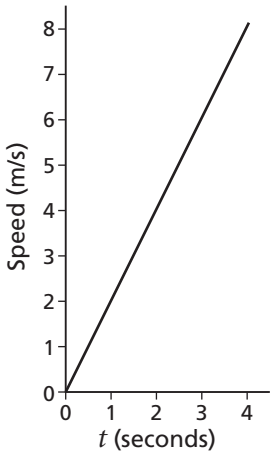
When a ball rolls down a slope it gets faster and faster. In fact, its speed (in m/s) is proportional to the time t (in seconds) it has been travelling.



In the example shown here, the speed of the ball is equal to $2t$.

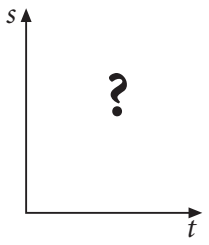
Here are a table of values and a graph showing how the speed increases over time.

t (seconds)	0	1	2	3	4
Speed (m/s)	0	2	4	6	8



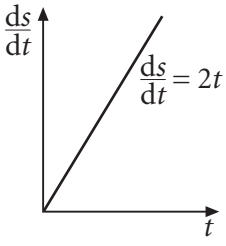
If s is the distance (in m) travelled in time t, then the speed of the ball is $\frac{ds}{dt}$.
So the graph above shows the gradient function for the distance.
The question arises: what does the distance function itself look like?

This is the reverse problem to finding a gradient function for a given function. We are given the gradient function or derivative and want to find the original function.



A1 Which function of t has the derivative $2t$?

Once we know the distance function we can work out, for example,
- how far the ball will travel in a given time
- how long it will take to go a given distance



A2 On a different slope the speed of the ball is $5t$.
What is the distance function in this case?

B Integration as the reverse of differentiation (answers p 187)

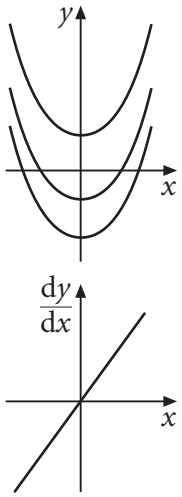
If you differentiate x^2 the result is the derivative, $2x$.

The reverse process is called **integration**.
Starting with $2x$ you ask: 'What function has derivative $2x$?'

D **B1** The obvious answer to the question above is x^2, but there is more to it than that.
Differentiate each of these functions: $x^2 + 1$, $x^2 + 5$, $x^2 - 9$, $x^2 + 30$
How would you answer the question: 'What function has $2x$ as its derivative?'

Because the derivative of any constant number, such as 5, is always zero, any function such as $x^2 + 5$, $x^2 + 7$, $x^2 - 3$, and so on, also has derivative $2x$.

So the function with derivative $2x$ is $x^2 + c$, where c can be any number.
This is illustrated in the graphs on the right.



K The process of going from $2x$ to $x^2 + c$ is called **integration**.
$x^2 + c$ is called the **indefinite integral** of $2x$
('indefinite' because c can be any number).
c is called the **constant of integration**.

B2 **(a)** Write down the derivative of $5x^2$.

(b) Hence write down the indefinite integral of $10x$.

B3 **(a)** Write down the derivative of each of these functions of x.

(i) $3x^2$ **(ii)** $4x^2$ **(iii)** $8x$ **(iv)** x^3 **(v)** $6x$

(b) Use your answers to (a) to write down the indefinite integral of

(i) $8x$ **(ii)** $3x^2$ **(iii)** 6 **(iv)** $6x$ **(v)** 8

D **B4** The derivative of x^4 is $4x^3$. Use this fact to find the indefinite integral of x^3.

Example 1

Find the indefinite integral of $12x^2$.

Solution

You are trying to find the function whose derivative is $12x^2$.
You know that the derivative of x^3 *is* $3x^2$.
So if you differentiate $4x^3$ *you will get* $12x^2$. The indefinite integral of $12x^2$ is $4x^3 + c$.

Exercise B (answers p 187)

1 Find the indefinite integrals of the following functions.

(a) $4x$ **(b)** $12x$ **(c)** $20x$ **(d)** x

2 Find the indefinite integral of

(a) $6x^2$ **(b)** $15x^2$ **(c)** x^2 **(d)** $2x^2$

3 Find the indefinite integrals of the following functions.
Check each answer by differentiating.

(a) $4x^3$ **(b)** $10x^4$ **(c)** $5x^2$ **(d)** $3x$

4 **(a)** Copy and complete this table of indefinite integrals.

Function	x	x^2	x^3	x^4
Indefinite integral				

(b) Use your results to help you write down a formula for the indefinite integral of the general function x^n, where n is a positive integer.

C Integrating polynomials (answers p 187)

The notation for 'the indefinite integral of $2x$' is $\int 2x\,\mathrm{d}x$ (read as 'integral $2x\,\mathrm{d}x$').
So we write $\int 2x\,\mathrm{d}x = x^2 + c$.
The reason for this notation will be explained in the next chapter. For now, think of $\int \mathrm{d}x$ as a single symbol with a blank space for the function to be integrated.

C1 Use the integral notation to write each of these statements.

(a) The indefinite integral of $3x^2$ is $x^3 + c$.

(b) The indefinite integral of $4x$ is $2x^2 + c$.

C2 Find the indefinite integral of $5x$ and write the statement 'the indefinite integral of $5x$ is …' using the integral notation.

C3 Repeat C2 for the function $6x^2$.

A rule for integrating a power of x emerged from the questions in exercise B.

K

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$$

In words, this rule says 'raise the index by 1 and divide by the new index'.
For example, $\int x^5\,\mathrm{d}x = \frac{x^6}{6} + c$.

Integrating a sum of functions

To differentiate the function $x^3 + x^2$ you differentiate each term separately and add, getting the derivative $3x^2 + 2x$.

It follows that $\int (3x^2 + 2x)\mathrm{d}x = x^3 + x^2 + c$.

So, as with differentiation, you integrate each term separately and add.

Integrating a multiple of a function

The derivative of $5x^2$ is 5 times the derivative of x^2.
The same applies to integration. For example,

$$\int 5x^2\,\mathrm{d}x = 5\int x^2\,\mathrm{d}x = 5\left(\frac{x^3}{3}\right) + c = \tfrac{5}{3}x^3 + c$$

Check that this is correct by differentiating $\frac{5}{3}x^3 + c$.

Integrating a polynomial

Using the rules given above, a polynomial can be integrated term by term.
If you are in any doubt about a result, differentiate it and check that you get the original function.

Example 2

If $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x^3 + 9x^2 - x + 3$, find y in terms of x.

Solution

y is the indefinite integral of $2x^3 + 9x^2 - x + 3$.

$$y = \int \left(2x^3 + 9x^2 - x + 3\right)\mathrm{d}x = 2\left(\frac{x^4}{4}\right) + 9\left(\frac{x^3}{3}\right) - \left(\frac{x^2}{2}\right) + 3x + c$$
$$= \tfrac{1}{2}x^4 + 3x^3 - \tfrac{1}{2}x^2 + 3x + c$$

Example 3

Find $\int (x + 3)(x^2 - 5)\,\mathrm{d}x$.

Solution

Multiply out the brackets to get a polynomial.

$$\int \left(x + 3\right)\left(x^2 - 5\right)\mathrm{d}x = \int \left(x^3 + 3x^2 - 5x - 15\right)\mathrm{d}x = \left(\frac{x^4}{4}\right) + 3\left(\frac{x^3}{3}\right) - 5\left(\frac{x^2}{2}\right) - 15x + c$$
$$= \tfrac{1}{4}x^4 + x^3 - \tfrac{5}{2}x^2 - 15x + c$$

Exercise C (answers p 188)

1 Find the following integrals.

(a) $\int x^3\,\mathrm{d}x$ **(b)** $\int 4x^2\,\mathrm{d}x$ **(c)** $\int 6x\,\mathrm{d}x$ **(d)** $\int 5x^4\,\mathrm{d}x$

2 Find y as a function of x for each of these.

(a) $\frac{dy}{dx} = x - 4$ **(b)** $\frac{dy}{dx} = 3x^2 + x$ **(c)** $\frac{dy}{dx} = x^2 + x + 1$ **(d)** $\frac{dy}{dx} = 5x^4 + 3$

3 Given that $f'(x) = 5x + 3x^3$, find an expression for $f(x)$.

4 Find the following integrals.

(a) $\int (2 - 3x + x^2)\,dx$ **(b)** $\int (5x^3 + 2x^5)\,dx$

5 Find y as a function of x for each of these.

(a) $\frac{dy}{dx} = 2x^3 - 7x + 3$ **(b)** $\frac{dy}{dx} = (x + 1)(x - 2)$

6 Find the indefinite integral of each of the following functions.

(a) $2(3x - 2)$ **(b)** $3x(x + 4)$ **(c)** $(2x - 1)^2$

7 Given that $f'(x) = (x + 2)(x - 1)(x + 4)$, find an expression for $f(x)$.

8 Find the following integrals.

(a) $\int (x + 2)(x - 5)\,dx$ **(b)** $\int x^2(2x + 1)\,dx$

D Finding the constant of integration

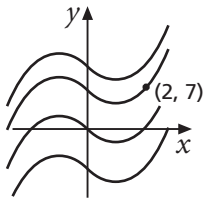
If you are told that $\frac{dy}{dx} = 3x^2 - 2$, then by integration it follows that $y = x^3 - 2x + c$.

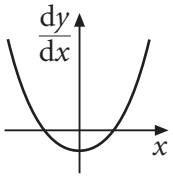
The equation $y = x^3 - 2x + c$ represents a family of graphs all having the same derivative, or gradient function.

If you are given the additional information that the graph goes through $(2, 7)$, then you can find the value of c:

$$7 = 2^3 - 2 \times 2 + c$$

$$\Rightarrow c = 3$$





Example 4

Given that $f'(x) = 8x^3 - 6x$ and that $f(2) = 9$, find $f(x)$ in terms of x.

Solution

First find the indefinite integral of $8x^3 - 6x$. $f(x) = \int (8x^3 - 6x)\,dx = 2x^4 - 3x^2 + c$

Now use the fact that $f(2) = 9$ *to find* c. $9 = 32 - 12 + c$

$\Rightarrow \quad c = -11$

So $f(x) = 2x^4 - 3x^2 - 11$

Exercise D (answers p 188)

1 Given that $f'(x) = 6x^2 + 4$ and that $f(1) = 7$, find an expression for $f(x)$.

2 Express y as a function of x for each of these.

(a) $\frac{dy}{dx} = 3x^2 + 4x$ and the (x, y) graph passes through $(1, 5)$.

(b) $\frac{dy}{dx} = x^2 + x + 1$ and the (x, y) graph passes through $(0, 3)$.

3 The curve C passes through the point $P\ (2, 1)$.
If $\frac{dy}{dx} = 2x - x^3$, find the equation of C.

4 Find an expression for $f(x)$ for each of these.

(a) $f'(x) = 5 - 2x$ and $f(2) = 4$ **(b)** $f'(x) = 3x(3x - 2)$ and $f(0) = 2$

5 (a) Given that $\frac{dy}{dx} = (x + 1)(2x - 3)$, find y as a function of x.

(b) If $y = 1$ when $x = 0$, find the value of y when $x = 3$.

6 The curve $y = f(x)$ passes through the points $(0, 5)$ and $(-2, k)$.
Given that $\frac{dy}{dx} = 4x + 3$, find the value of k.

7 The rate of growth of a population of micro-organisms is modelled by the equation

$$\frac{dP}{dt} = 3t^2 + 6t$$

where P is the population size at time t hours.
Given that $P = 100$ when $t = 1$, find P in terms of t.

8 The equation of a curve is $y = f(x)$. The curve goes through the points $(1, 3)$ and $(3, 7)$. Given that $f'(x) = 4x + p$, where p is a number, find

(a) the value of p

(b) the equation of the curve

***9** The curve with equation $y = f(x)$ goes through the points $(0, 5)$, $(1, 11)$ and $(2, 37)$. Given that $f'(x) = ax^2 + bx$, find

(a) the values of a, b and c (the constant of integration)

(b) the equation of the curve

Key points

- Integration is the reverse of differentiation. (p 137)
- The indefinite integral of a function includes a constant term. (p 137)
- The indefinite integral of a function f(x) is denoted by $\int \mathrm{f}(x)\,\mathrm{d}x$. (p 138)
- $\int x^n\,\mathrm{d}x = \dfrac{x^{n+1}}{n+1} + c$ for positive integers n. (p 138)
- The indefinite integral of a sum of functions is the sum of the separate indefinite integrals.
  The indefinite integral of k times a function is k times the indefinite integral of the function. (p 139)
- Given $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ (or f′(x)) and the value of y (or f(x)) for a given value of x, the value of the constant of integration can be found. (p 140)

Mixed questions (answers p 188)

1 Find

(a) $\int (2x^2 + 3x - 1)\,\mathrm{d}x$ **(b)** $\int x(5x^4 + 2)\,\mathrm{d}x$ **(c)** $\int (2x^2 + 3)(x - 4)\,\mathrm{d}x$

2 Given that $\mathrm{f}'(x) = x^3 + 12x^2 - 2$ and $\mathrm{f}(2) = 0$, find f(x) in terms of x.

3 The curve C goes through the point (3, 5).
The gradient $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point (x, y) on C is given by the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^2 - 3x$.
Find the equation of C.

4 Given that $\dfrac{\mathrm{d}s}{\mathrm{d}t} = (t + 3)(t - 1)$ and that $s = 10$ when $t = 3$, find s in terms of t.

5 Given that $\mathrm{f}'(x) = 3x^2 + ax$, $\mathrm{f}(-2) = 8$ and $\mathrm{f}(1) = 2$, find

(a) the value of a

(b) an expression for f(x) in terms of x

6 The rate of growth of a bird population is modelled by the equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = a + bt$$

where P is the population at time t, and a and b are constants.
Given that $P = 100$ when $t = 0$, $P = 172$ when $t = 4$, and $P = 202$ when $t = 6$,

(a) find the formula for P in terms of t

(b) find the values of t for which $P = 250$

***7** The rate of spread of an illness affecting animals in a colony is modelled by the equation

$$\frac{dN}{dt} = 5 - \tfrac{4}{9}t - \tfrac{1}{9}t^2$$

where N is the number of animals affected and t is the time in weeks since recording began.

When recording began, 36 animals were affected.

(a) Find a formula for N in terms of t.

(b) How many affected animals are there after 3 weeks?

(c) After how many weeks does the number of affected animals reach a maximum?

(d) Show that, according to the model, the number of affected animals decreases after reaching a maximum and is zero when $t = 12$.

Test yourself (answers p 188)

None of these questions requires a calculator.

1 Find

(a) $\int (x^3 + 2x^2 - x)\,dx$ **(b)** $\int (x^4 + 7x - 1)\,dx$ **(c)** $\int (x^8 + 5x^6)\,dx$

2 Find

(a) $\int 3x^2(x - 2)\,dx$ **(b)** $\int (4x + 1)(3x^2 - 1)\,dx$ **(c)** $\int (3x - 2)^2\,dx$

3 The gradient function of a curve is given by $\frac{dy}{dx} = 6x^2 - 1$.

The curve goes through the point $(-1, 4)$. Find the equation of the curve.

4 Given that $\frac{dy}{dx} = (6x + 5)(x - 1)$ and that $y = 0$ when $x = 1$, find y in terms of x.

5 Given that $f'(x) = 10x^4 - 12x^3 - 4$ and $f(2) = 10$, find

(a) $f(x)$ in terms of x **(b)** $f(-2)$

6 Given that $f'(x) = (x + 1)(3x - 5)$ and $f(3) = 6$, find $f(x)$ in terms of x.

7 The curve C goes through the point $(-2, 2)$.

The gradient function of C is given by $\frac{dy}{dx} = (3x - 1)^2$.

Find the equation of C.