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# 10 Integration

In this chapter you will learn

- what integration is and how it is related to differentiation
- how to integrate polynomial functions

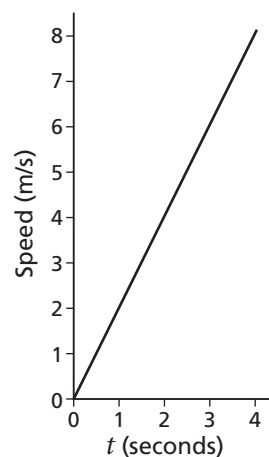
## A Thinking backwards (answers p 187)

When a ball rolls down a slope it gets faster and faster. In fact, its speed (in m/s) is proportional to the time  $t$  (in seconds) it has been travelling.

In the example shown here, the speed of the ball is equal to  $2t$ .

Here are a table of values and a graph showing how the speed increases over time.

$t$ (seconds)	0	1	2	3	4
Speed (m/s)	0	2	4	6	8



If  $s$  is the distance (in m) travelled in time  $t$ , then the speed of the ball is  $\frac{ds}{dt}$ . So the graph above shows the gradient function for the distance.

The question arises: what does the distance function itself look like?

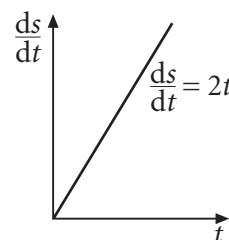
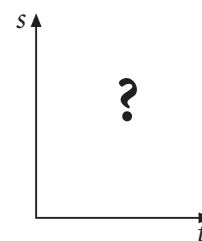
This is the reverse problem to finding a gradient function for a given function. We are given the gradient function or derivative and want to find the original function.

**A1** Which function of  $t$  has the derivative  $2t$ ?

Once we know the distance function we can work out, for example,

- how far the ball will travel in a given time
- how long it will take to go a given distance

**A2** On a different slope the speed of the ball is  $5t$ . What is the distance function in this case?



## B Integration as the reverse of differentiation (answers p 187)

If you differentiate  $x^2$  the result is the derivative,  $2x$ .

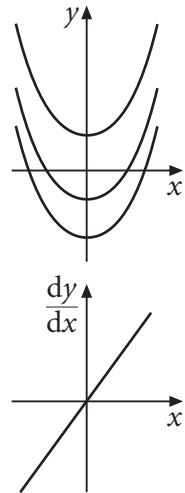
The reverse process is called **integration**.

Starting with  $2x$  you ask: 'What function has derivative  $2x$ ?'

- D B1** The obvious answer to the question above is  $x^2$ , but there is more to it than that. Differentiate each of these functions:  $x^2 + 1$ ,  $x^2 + 5$ ,  $x^2 - 9$ ,  $x^2 + 30$   
How would you answer the question: 'What function has  $2x$  as its derivative?'

Because the derivative of any constant number, such as 5, is always zero, any function such as  $x^2 + 5$ ,  $x^2 + 7$ ,  $x^2 - 3$ , and so on, also has derivative  $2x$ .

So the function with derivative  $2x$  is  $x^2 + c$ , where  $c$  can be any number. This is illustrated in the graphs on the right.



- K** The process of going from  $2x$  to  $x^2 + c$  is called **integration**.  $x^2 + c$  is called the **indefinite integral** of  $2x$  ('indefinite' because  $c$  can be any number).  $c$  is called the **constant of integration**.

**B2 (a)** Write down the derivative of  $5x^2$ .

(b) Hence write down the indefinite integral of  $10x$ .

**B3 (a)** Write down the derivative of each of these functions of  $x$ .

(i)  $3x^2$

(ii)  $4x^2$

(iii)  $8x$

(iv)  $x^3$

(v)  $6x$

(b) Use your answers to (a) to write down the indefinite integral of

(i)  $8x$

(ii)  $3x^2$

(iii)  $6$

(iv)  $6x$

(v)  $8$

- D B4** The derivative of  $x^4$  is  $4x^3$ . Use this fact to find the indefinite integral of  $x^3$ .

---

### Example 1

Find the indefinite integral of  $12x^2$ .

#### Solution

*You are trying to find the function whose derivative is  $12x^2$ .*

*You know that the derivative of  $x^3$  is  $3x^2$ .*

*So if you differentiate  $4x^3$  you will get  $12x^2$ .*

The indefinite integral of  $12x^2$  is  $4x^3 + c$ .

---

**Exercise B** (answers p 187)

1 Find the indefinite integrals of the following functions.

- (a)  $4x$                       (b)  $12x$                       (c)  $20x$                       (d)  $x$

2 Find the indefinite integral of

- (a)  $6x^2$                       (b)  $15x^2$                       (c)  $x^2$                       (d)  $2x^2$

3 Find the indefinite integrals of the following functions.

Check each answer by differentiating.

- (a)  $4x^3$                       (b)  $10x^4$                       (c)  $5x^2$                       (d)  $3x$

4 (a) Copy and complete this table of indefinite integrals.

Function	$x$	$x^2$	$x^3$	$x^4$
Indefinite integral				

(b) Use your results to help you write down a formula for the indefinite integral of the general function  $x^n$ , where  $n$  is a positive integer.

**C Integrating polynomials** (answers p 187)

The notation for ‘the indefinite integral of  $2x$ ’ is  $\int 2x \, dx$  (read as ‘integral  $2x \, dx$ ’).

So we write  $\int 2x \, dx = x^2 + c$ .

The reason for this notation will be explained in the next chapter. For now, think of  $\int \, dx$  as a single symbol with a blank space for the function to be integrated.

**C1** Use the integral notation to write each of these statements.

- (a) The indefinite integral of  $3x^2$  is  $x^3 + c$ .  
(b) The indefinite integral of  $4x$  is  $2x^2 + c$ .

**C2** Find the indefinite integral of  $5x$  and write the statement ‘the indefinite integral of  $5x$  is ...’ using the integral notation.

**C3** Repeat C2 for the function  $6x^2$ .

A rule for integrating a power of  $x$  emerged from the questions in exercise B.

**K** 
$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

In words, this rule says ‘raise the index by 1 and divide by the new index’.

For example,  $\int x^5 \, dx = \frac{x^6}{6} + c$ .

### Integrating a sum of functions

To differentiate the function  $x^3 + x^2$  you differentiate each term separately and add, getting the derivative  $3x^2 + 2x$ .

It follows that  $\int(3x^2 + 2x)dx = x^3 + x^2 + c$ .

So, as with differentiation, you integrate each term separately and add.

### Integrating a multiple of a function

The derivative of  $5x^2$  is 5 times the derivative of  $x^2$ .

The same applies to integration. For example,

$$\int 5x^2 dx = 5 \int x^2 dx = 5 \left( \frac{x^3}{3} \right) + c = \frac{5}{3}x^3 + c$$

Check that this is correct by differentiating  $\frac{5}{3}x^3 + c$ .

### Integrating a polynomial

Using the rules given above, a polynomial can be integrated term by term.

If you are in any doubt about a result, differentiate it and check that you get the original function.

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#### Example 2

If  $\frac{dy}{dx} = 2x^3 + 9x^2 - x + 3$ , find  $y$  in terms of  $x$ .

#### Solution

$y$  is the indefinite integral of  $2x^3 + 9x^2 - x + 3$ .

$$\begin{aligned} y &= \int(2x^3 + 9x^2 - x + 3) dx = 2 \left( \frac{x^4}{4} \right) + 9 \left( \frac{x^3}{3} \right) - \left( \frac{x^2}{2} \right) + 3x + c \\ &= \frac{1}{2}x^4 + 3x^3 - \frac{1}{2}x^2 + 3x + c \end{aligned}$$

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#### Example 3

Find  $\int(x + 3)(x^2 - 5) dx$ .

#### Solution

*Multiply out the brackets to get a polynomial.*

$$\begin{aligned} \int(x + 3)(x^2 - 5) dx &= \int(x^3 + 3x^2 - 5x - 15) dx = \left( \frac{x^4}{4} \right) + 3 \left( \frac{x^3}{3} \right) - 5 \left( \frac{x^2}{2} \right) - 15x + c \\ &= \frac{1}{4}x^4 + x^3 - \frac{5}{2}x^2 - 15x + c \end{aligned}$$

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#### Exercise C (answers p 188)

1 Find the following integrals.

(a)  $\int x^3 dx$

(b)  $\int 4x^2 dx$

(c)  $\int 6x dx$

(d)  $\int 5x^4 dx$

2 Find  $y$  as a function of  $x$  for each of these.

(a)  $\frac{dy}{dx} = x - 4$       (b)  $\frac{dy}{dx} = 3x^2 + x$       (c)  $\frac{dy}{dx} = x^2 + x + 1$       (d)  $\frac{dy}{dx} = 5x^4 + 3$

3 Given that  $f'(x) = 5x + 3x^3$ , find an expression for  $f(x)$ .

4 Find the following integrals.

(a)  $\int (2 - 3x + x^2) dx$       (b)  $\int (5x^3 + 2x^5) dx$

5 Find  $y$  as a function of  $x$  for each of these.

(a)  $\frac{dy}{dx} = 2x^3 - 7x + 3$       (b)  $\frac{dy}{dx} = (x + 1)(x - 2)$

6 Find the indefinite integral of each of the following functions.

(a)  $2(3x - 2)$       (b)  $3x(x + 4)$       (c)  $(2x - 1)^2$

7 Given that  $f'(x) = (x + 2)(x - 1)(x + 4)$ , find an expression for  $f(x)$ .

8 Find the following integrals.

(a)  $\int (x + 2)(x - 5) dx$       (b)  $\int x^2(2x + 1) dx$

## D Finding the constant of integration

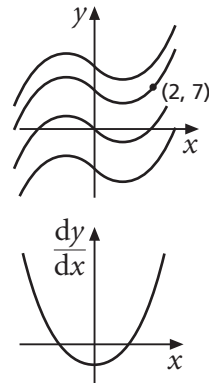
If you are told that  $\frac{dy}{dx} = 3x^2 - 2$ , then by integration it follows that  $y = x^3 - 2x + c$ .

The equation  $y = x^3 - 2x + c$  represents a family of graphs all having the same derivative, or gradient function.

If you are given the additional information that the graph goes through  $(2, 7)$ , then you can find the value of  $c$ :

$$7 = 2^3 - 2 \times 2 + c$$

$$\Rightarrow c = 3$$



### Example 4

Given that  $f'(x) = 8x^3 - 6x$  and that  $f(2) = 9$ , find  $f(x)$  in terms of  $x$ .

#### Solution

First find the indefinite integral of  $8x^3 - 6x$ .       $f(x) = \int (8x^3 - 6x) dx = 2x^4 - 3x^2 + c$

Now use the fact that  $f(2) = 9$  to find  $c$ .       $9 = 32 - 12 + c$

$$\Rightarrow c = -11$$

$$\text{So } f(x) = 2x^4 - 3x^2 - 11$$

**Exercise D** (answers p 188)

- 1** Given that  $f'(x) = 6x^2 + 4$  and that  $f(1) = 7$ , find an expression for  $f(x)$ .
- 2** Express  $y$  as a function of  $x$  for each of these.
- (a)  $\frac{dy}{dx} = 3x^2 + 4x$  and the  $(x, y)$  graph passes through  $(1, 5)$ .
- (b)  $\frac{dy}{dx} = x^2 + x + 1$  and the  $(x, y)$  graph passes through  $(0, 3)$ .
- 3** The curve  $C$  passes through the point  $P(2, 1)$ .  
If  $\frac{dy}{dx} = 2x - x^3$ , find the equation of  $C$ .
- 4** Find an expression for  $f(x)$  for each of these.
- (a)  $f'(x) = 5 - 2x$  and  $f(2) = 4$       (b)  $f'(x) = 3x(3x - 2)$  and  $f(0) = 2$
- 5** (a) Given that  $\frac{dy}{dx} = (x + 1)(2x - 3)$ , find  $y$  as a function of  $x$ .  
(b) If  $y = 1$  when  $x = 0$ , find the value of  $y$  when  $x = 3$ .
- 6** The curve  $y = f(x)$  passes through the points  $(0, 5)$  and  $(-2, k)$ .  
Given that  $\frac{dy}{dx} = 4x + 3$ , find the value of  $k$ .
- 7** The rate of growth of a population of micro-organisms is modelled by the equation  
$$\frac{dP}{dt} = 3t^2 + 6t$$
where  $P$  is the population size at time  $t$  hours.  
Given that  $P = 100$  when  $t = 1$ , find  $P$  in terms of  $t$ .
- 8** The equation of a curve is  $y = f(x)$ . The curve goes through the points  $(1, 3)$  and  $(3, 7)$ . Given that  $f'(x) = 4x + p$ , where  $p$  is a number, find
- (a) the value of  $p$   
(b) the equation of the curve
- \*9** The curve with equation  $y = f(x)$  goes through the points  $(0, 5)$ ,  $(1, 11)$  and  $(2, 37)$ .  
Given that  $f'(x) = ax^2 + bx$ , find
- (a) the values of  $a$ ,  $b$  and  $c$  (the constant of integration)  
(b) the equation of the curve

### Key points

- Integration is the reverse of differentiation. (p 137)
- The indefinite integral of a function includes a constant term. (p 137)
- The indefinite integral of a function  $f(x)$  is denoted by  $\int f(x) dx$ . (p 138)
- $\int x^n dx = \frac{x^{n+1}}{n+1} + c$  for positive integers  $n$ . (p 138)
- The indefinite integral of a sum of functions is the sum of the separate indefinite integrals.  
The indefinite integral of  $k$  times a function is  $k$  times the indefinite integral of the function. (p 139)
- Given  $\frac{dy}{dx}$  (or  $f'(x)$ ) and the value of  $y$  (or  $f(x)$ ) for a given value of  $x$ , the value of the constant of integration can be found. (p 140)

### Mixed questions (answers p 188)

1 Find

(a)  $\int (2x^2 + 3x - 1) dx$       (b)  $\int x(5x^4 + 2) dx$       (c)  $\int (2x^2 + 3)(x - 4) dx$

2 Given that  $f'(x) = x^3 + 12x^2 - 2$  and  $f(2) = 0$ , find  $f(x)$  in terms of  $x$ .

3 The curve  $C$  goes through the point  $(3, 5)$ .

The gradient  $\frac{dy}{dx}$  at the point  $(x, y)$  on  $C$  is given by the equation  $\frac{dy}{dx} = \frac{1}{2}x^2 - 3x$ .

Find the equation of  $C$ .

4 Given that  $\frac{ds}{dt} = (t + 3)(t - 1)$  and that  $s = 10$  when  $t = 3$ , find  $s$  in terms of  $t$ .

5 Given that  $f'(x) = 3x^2 + ax$ ,  $f(-2) = 8$  and  $f(1) = 2$ , find

- (a) the value of  $a$   
(b) an expression for  $f(x)$  in terms of  $x$

6 The rate of growth of a bird population is modelled by the equation

$$\frac{dP}{dt} = a + bt$$

where  $P$  is the population at time  $t$ , and  $a$  and  $b$  are constants.

Given that  $P = 100$  when  $t = 0$ ,  $P = 172$  when  $t = 4$ , and  $P = 202$  when  $t = 6$ ,

- (a) find the formula for  $P$  in terms of  $t$   
(b) find the values of  $t$  for which  $P = 250$

- \*7 The rate of spread of an illness affecting animals in a colony is modelled by the equation

$$\frac{dN}{dt} = 5 - \frac{4}{9}t - \frac{1}{9}t^2$$

where  $N$  is the number of animals affected and  $t$  is the time in weeks since recording began.

When recording began, 36 animals were affected.

- Find a formula for  $N$  in terms of  $t$ .
- How many affected animals are there after 3 weeks?
- After how many weeks does the number of affected animals reach a maximum?
- Show that, according to the model, the number of affected animals decreases after reaching a maximum and is zero when  $t = 12$ .

### Test yourself (answers p 188)

None of these questions requires a calculator.

1 Find

(a)  $\int (x^3 + 2x^2 - x) dx$       (b)  $\int (x^4 + 7x - 1) dx$       (c)  $\int (x^8 + 5x^6) dx$

2 Find

(a)  $\int 3x^2(x - 2) dx$       (b)  $\int (4x + 1)(3x^2 - 1) dx$       (c)  $\int (3x - 2)^2 dx$

3 The gradient function of a curve is given by  $\frac{dy}{dx} = 6x^2 - 1$ .

The curve goes through the point  $(-1, 4)$ . Find the equation of the curve.

4 Given that  $\frac{dy}{dx} = (6x + 5)(x - 1)$  and that  $y = 0$  when  $x = 1$ , find  $y$  in terms of  $x$ .

5 Given that  $f'(x) = 10x^4 - 12x^3 - 4$  and  $f(2) = 10$ , find

(a)  $f(x)$  in terms of  $x$       (b)  $f(-2)$

6 Given that  $f'(x) = (x + 1)(3x - 5)$  and  $f(3) = 6$ , find  $f(x)$  in terms of  $x$ .

7 The curve  $C$  goes through the point  $(-2, 2)$ .

The gradient function of  $C$  is given by  $\frac{dy}{dx} = (3x - 1)^2$ .

Find the equation of  $C$ .